Exercise (see "wine seller" for solution structure):

In the Principal-Agent model, assume that the agents´ utilities are given by $U = \theta_i q_i - t_i$, and that the principal has the utility function $W = t_i - S(q_i)$, for i=1,2,3,4 with $S' > 0$ and $S'' < 0$. Additionally, assume that θ may take four values, $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, with $\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_4 = \Delta\theta$ and respective probabilities v_1, v_2, v_3, v_4 such that $v_1 + v_2 + v_3 + v_4 = 1$. A direct revelation mechanism has the form $\{(t_1, q_1), (t_2, q_2), (t_3, q_3), (t_4, q_4)\}$.

Find:

a) The incentive restrictions for each agent.

b) The implementation conditions.

c) Write the principal´s optimization problem with the necessary restrictions.

d) Find the second-best results in terms of $S'(q_i)$.

e) The first-best result in terms of $S'(q_i)$,

f) Compare the second-best results with the first-best results.

Solution:

a) Incentive restrictions for each agent.

Agent 1

$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2$

$\theta_1 q_1 - t_1 \geq \theta_1 q_3 - t_3$

$\theta_1 q_1 - t_1 \geq \theta_1 q_4 - t_4$

Agent 2

$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1$

$\theta_2 q_2 - t_2 \geq \theta_2 q_3 - t_3$

$\theta_2 q_2 - t_2 \geq \theta_2 q_4 - t_4$

Agent 3

$\theta_3 q_3 - t_3 \geq \theta_3 q_1 - t_1$

$\theta_3 q_3 - t_3 \geq \theta_3 q_2 - t_2$

$\theta_3 q_3 - t_3 \geq \theta_3 q_4 - t_4$

Agent 4

$\theta_4 q_4 - t_4 \geq \theta_4 q_1 - t_1$

$\theta_4 q_4 - t_4 \geq \theta_4 q_2 - t_2$

$\theta_4 q_4 - t_4 \geq \theta_4 q_3 - t_3$

b) Implementation conditions:

$\theta_i q_i - t_i \geq 0$ i= 1,2,3,4.

c) Principal´s optimization problem with the necessary restrictions:

$$\max_{t,q} \sum_{i=1}^{4} v_i\left(t_i - S(q_i)\right)$$

subject to the restrictions found in (a) and (b)

d) Second-best results in terms of $S'(q_i)$

Assume $\theta_1 < \theta_2 < \theta_3 < \theta_4$ with $\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_4 = \Delta\theta$

Consider the common results:

1.- The highest type has an efficient allocation: $q_4 = q_4*$

2.- All the remaining types have a sub-efficient allocation: $q_1 < q_1*, q_2 < q_2*, q_3 < q_3*$

3.- All types (with the exception of the lowest type) remain indifferent in their contracts:

$\theta_2 q_2 - t_2 = \theta_2 q_1 - t_1 \Rightarrow t_2 = \theta_2 q_2 - \theta_2 q_1 + t_1$ (1)

$\theta_3 q_3 - t_3 = \theta_3 q_2 - t_2 \Rightarrow t_3 = \theta_3 q_3 - \theta_3 q_2 + t_2$ (2)

$\theta_4 q_4 - t_4 = \theta_4 q_3 - t_3 \Rightarrow t_4 = \theta_4 q_4 - \theta_4 q_3 + t_3$ (3)

4.- The lowest type obtains zero surplus: $U = \theta_1 q_1 - t_1 = 0 \Rightarrow t_1 = \theta_1 q_1$

5.- Utilities increases with the type (with the exception of the lowest type):

$0 = U_1 < U_2 < U_3 < U_4$

Using $t_1 = \theta_1 q_1$ in (1)', (2)' y (3)':

$t_1 = \theta_1 q_1$

$t_2 = \theta_1 q_1 - \theta_2 q_1 + \theta_2 q_2 = \Delta\theta q_1 + \theta_2 q_2$

$t_3 = \Delta\theta\, q_1 + \theta_2 q_2 - \theta_3 q_2 + \theta_3 q_3 = \Delta\theta q_1 + \Delta\theta\, q_2 + \theta_3 q_3$

$t_4 = \Delta\theta\, q_1 + \Delta\theta\, q_2 + \theta_3 q_3 - \theta_4 q_3 + \theta_4 q_4 = \Delta\theta q_1 + \Delta\theta\, q_2 + \Delta\theta\, q_3 + \theta_4 q_4$

In the problem:

$$\max \quad v_1\left(\theta_1 q_1 - S(q_1)\right) + v_2\left(\Delta\theta q_1 + \theta_2 q_2 - S(q_2)\right) + v_3\left(\Delta\theta q_1 + \Delta\theta q_2 + \theta_3 q_3 - S(q_3)\right) + v_4\left(\Delta\theta q_1 + \Delta\theta q_2 + \Delta\theta q_3 + \theta_4 q_4 - S(q_4)\right)$$

$$\frac{\partial(\cdot)}{\partial(q_1)} = v_1\left[\theta_1 - s'(q_1)\right] + v_2\Delta\theta + v_3\Delta\theta + v_4\Delta\theta = 0 \Rightarrow s'(q_1) = \Delta\theta\frac{v_2 + v_3 + v_4}{v_1} + \theta_1$$

$$\frac{\partial(\cdot)}{\partial(q_2)} = v_2\left[\theta_2 - s'(q_2)\right] + v_3\Delta\theta + v_4\Delta\theta = 0 \Rightarrow s'(q_2) = \Delta\theta\frac{v_3 + v_4}{v_2} + \theta_2$$

$$\frac{\partial(\cdot)}{\partial(q_3)} = v_3\left[\theta_3 - s'(q_3)\right] + v_4\Delta\theta = 0 \Rightarrow s'(q_3) = \Delta\theta\frac{v_4}{v_3} + \theta_3$$

$$\frac{\partial(\cdot)}{\partial(q_4)} = v_4\left[\theta_4 - s'(q_4)\right] = 0 \Rightarrow s'(q_4) = \theta_4$$

e) First best result in terms of $S'(q_i)$,

$$\max_{t,q} \sum_{i=1}^{4} t_i - S(q_i)$$

s.t.

$$\theta_i q_i - t_i \geq 0$$

Since there is perfect information the surplus of each consumer will be 0: $\theta_i q_i - t_i = 0$

$$\max_{q} \sum_{i=1}^{4} \theta_i q_i - S(q_i)$$

$$\frac{\partial(\cdot)}{\partial(q_i)} = \theta_i - s'(q_i) = 0 \Rightarrow s'(q_i) = \theta_i$$

Which implies

$q_i = q_i *$

$t_i * = \theta_i q_i *$ $\forall$i= 1,2,3,4.

f) Comparison of first-best and second-best results:

The highest type has the efficient allocations: $s'(q_4) = \theta_4$ and $q_4 = q_4 *$

The other types have sub-efficient allocations:

$$s'(q_1) = \Delta\theta \frac{v_2 + v_3 + v_4}{v_1} + \theta_1 < \theta_1 = s'(q_1 *)\ ;\ \Rightarrow q_1 < q_1 *$$

$$s'(q_2) = \Delta\theta \frac{v_3 + v_4}{v_2} + \theta_2 < \theta_2 = s'(q_2 *)\ ;\ \Rightarrow q_2 < q_2 *$$

$$s'(q_3) = \Delta\theta \frac{v_4}{v_3} + \theta_3 < \theta_3 = s'(q_3 *)\ ;\ \Rightarrow q_3 < q_3 *$$

Additionally, $0 = U_1 < U_2 < U_3 < U_4$

Therefore, the second-best allocations are less than the first-best allocations, with the exception of type 4 who gets the same allocation.