A Note on Bargaining over Complementary Pieces of Information in Networks

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This appendix provides the details of the applications of Cramer's rule used in the proofs of propositions 1 and 2.

Proof of Proposition 1

(a) Note that

$$|A| = \begin{vmatrix} (1+\delta) & 1 & \dots & 1 \\ 1 & (1+\delta) & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & (1+\delta) \end{vmatrix} = \begin{vmatrix} (n-1+\delta) & 1 & \dots & 1 \\ (n-1+\delta) & (1+\delta) & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ (n-1+\delta) & 1 & \dots & (1+\delta) \end{vmatrix}$$

$$= (n-1+\delta) \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & (1+\delta) & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & (1+\delta) \end{vmatrix} = (n-1+\delta) \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & \delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & \delta \end{vmatrix}$$

$$= \delta^{n-2}(n-1+\delta),$$

so that $A \cdot q^* = b$ has a unique solution. Without loss of generality, let us compute the solution q^*_{21}. We need to compute the determinant of the matrix obtained by replacing the first column of matrix A by the vector with components $(v_1^{gs} - \delta v_j^{gs} + n - 1 + \delta)/2$. Let us denote this matrix by Ab_1. To ease notation, let us write b_j to indicate each $(v_1^{gs} - \delta v_j^{gs} + n - 1 + \delta)/2$, $j \neq 1$, so that $b = (b_2, \dots, b_n)$. Then, notice that

$$|Ab_1| = \begin{vmatrix} b_2 & 1 & 1 & \dots & 1 \\ b_3 & (1+\delta) & 1 & \dots & 1 \\ b_4 & 1 & (1+\delta) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & 1 & 1 & \dots & (1+\delta) \end{vmatrix} = \begin{vmatrix} b_2 & 0 & 0 & \dots & 1 \\ b_3 & \delta & 0 & \dots & 1 \\ b_4 & 0 & \delta & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & -\delta & -\delta & \dots & (1+\delta) \end{vmatrix}$$

$$= \begin{vmatrix} b_2 & 0 & 0 & \dots & 1 \\ b_3 - b_2 & \delta & 0 & \dots & 0 \\ b_4 - b_2 & 0 & \delta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n - b_2 & -\delta & -\delta & \dots & \delta \end{vmatrix} = b_2 \begin{vmatrix} \delta & 0 & \dots & 0 \\ 0 & \delta & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\delta & -\delta & \dots & \delta \end{vmatrix} + (-1)^n \begin{vmatrix} b_3 - b_2 & \delta & \dots & 0 \\ b_4 - b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n - b_2 & -\delta & \dots & -\delta \end{vmatrix}$$

$$= b_2\delta^{n-2} + (-1)^n \begin{vmatrix} b_3 - b_2 & \delta & \dots & 0 \\ b_4 - b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n - b_2 & -\delta & \dots & -\delta \end{vmatrix}.$$

As for the last determinant in the expression above, note that

$$\begin{vmatrix} b_3 - b_2 & \delta & 0 & 0 & \dots & 0 \\ b_4 - b_2 & 0 & \delta & 0 & \dots & 0 \\ b_4 - b_2 & 0 & 0 & \delta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} - b_2 & 0 & 0 & 0 & \dots & \delta \\ b_n - b_2 & -\delta & -\delta & -\delta & \dots & -\delta \end{vmatrix} = \begin{vmatrix} b_3 - b_2 & \delta & 0 & 0 & \dots & 0 \\ b_4 - b_2 & 0 & \delta & 0 & \dots & 0 \\ b_5 - b_2 & 0 & 0 & \delta & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} - b_2 & 0 & 0 & 0 & \dots & \delta \\ \sum_{k=3}^{n}(b_k - b_2) & 0 & 0 & 0 & \dots & 0 \end{vmatrix}$$

$$= (-1)^{n-1}\delta^{n-3}\sum_{k=3}^{n}(b_k - b_2).$$

Then, using the expression of each b_j, $j \neq 1$, we obtain

$$\begin{aligned} |Ab_1| &= \delta^{n-2}(v_1^{g_{\mathrm{S}}} - \delta v_2^{g_{\mathrm{S}}} + n - 1 + \delta)/2 + (-1)^{2n-1}\delta^{n-2}\sum_{k=3}^{n}(v_2^{g_{\mathrm{S}}} - v_k^{g_{\mathrm{S}}})/2 \\ &= \frac{\delta^{n-2}}{2}\left(v_1^{g_{\mathrm{S}}} - (n - 2 + \delta)v_2^{g_{\mathrm{S}}} + \sum_{k=3}^{n} v_k^{g_{\mathrm{S}}} + (n - 1 + \delta)\right). \end{aligned}$$

Application of Cramer's rule gives us

$$q_{21}^* = |Ab_1| \,/\, |A| = \frac{1}{2} + \frac{v_1^{g_{\mathrm{S}}} - (n - 2 + \delta)v_2^{g_{\mathrm{S}}} + \sum_{k=3}^{n} v_k^{g_{\mathrm{S}}}}{2(n - 1 + \delta)}.$$

To compute any relative price q_{j1}^*, $j \neq 1$, one must follow an argument analogous to the one above. Hence, there is a unique SPE s^* for the game $\Gamma(g_{\mathrm{S}}, M)$ such that any peripheral agent $j \neq 1$ charges a relative price

$$q_{j1}^* = \frac{1}{2} + \frac{v_1^{g_{\mathrm{S}}} - (n - 2 + \delta)v_j^{g_{\mathrm{S}}} + \sum_{k \neq 1, j} v_k^{g_{\mathrm{S}}}}{2(n - 1 + \delta)}$$

to the central agent (agent 1) in the star network.

(b) Note that

$$|A| = \begin{vmatrix} (1+\delta) & \delta & \dots & \delta \\ \delta & (1+\delta) & \dots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ \delta & \delta & \dots & (1+\delta) \end{vmatrix} = \begin{vmatrix} [1+(n-1)\delta] & \delta & \dots & \delta \\ [1+(n-1)\delta] & (1+\delta) & \dots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ [1+(n-1)\delta] & \delta & \dots & (1+\delta) \end{vmatrix}$$

$$= [1+(n-1)\delta] \begin{vmatrix} 1 & \delta & \dots & \delta \\ 1 & (1+\delta) & \dots & \delta \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \delta & \dots & (1+\delta) \end{vmatrix} = [1+(n-1)\delta] \begin{vmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 1 \end{vmatrix}$$

$$= [1+(n-1)\delta],$$

so that $A \cdot q^{**} = b$ has a unique solution. Without loss of generality, let us compute the solution q_{21}^{**}. We need to compute the determinant of the matrix obtained by replacing the first column of matrix A by the vector with components $(\delta v_1^{gs} - v_j^{gs} + 1 + (n-1)\delta)/2$. Let us denote this matrix by Ab_1. To ease notation, let us write b_j to indicate each $(v_1^{gs} - \delta v_j^{gs} + n - 1 + \delta)/2$, $j \neq 1$, so that $b = (b_2, \dots, b_n)$. Then, notice that

$$|Ab_1| = \begin{vmatrix} b_2 & \delta & \delta & \dots & \delta \\ b_3 & (1+\delta) & \delta & \dots & \delta \\ b_4 & 1 & (1+\delta) & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & \delta & \delta & \dots & (1+\delta) \end{vmatrix} = \begin{vmatrix} b_2 & 0 & 0 & \dots & \delta \\ b_3 & 1 & 0 & \dots & \delta \\ b_4 & 0 & 1 & \dots & \delta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & -1 & -1 & \dots & (1+\delta) \end{vmatrix}$$

$$= \begin{vmatrix} b_2 & 0 & 0 & \dots & \delta \\ b_3 - b_2 & 1 & 0 & \dots & 0 \\ b_4 - b_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n - b_2 & -1 & -1 & \dots & 1 \end{vmatrix} = b_2 \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 1 \end{vmatrix} + (-1)^n \delta \begin{vmatrix} b_3 - b_2 & 1 & \dots & 0 \\ b_4 - b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n - b_2 & -1 & \dots & -1 \end{vmatrix}$$

$$= b_2 + (-1)^n \delta \begin{vmatrix} b_3 - b_2 & 1 & \dots & 0 \\ b_4 - b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n - b_2 & -1 & \dots & -1 \end{vmatrix}.$$

As for the last determinant in the expression above, note that

$$\begin{vmatrix} b_3 - b_2 & 1 & 0 & 0 & \dots & 0 \\ b_4 - b_2 & 0 & 1 & 0 & \dots & 0 \\ b_4 - b_2 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} - b_2 & 0 & 0 & 0 & \dots & 1 \\ b_n - b_2 & -1 & -1 & -1 & \dots & -1 \end{vmatrix} = \begin{vmatrix} b_3 - b_2 & 1 & 0 & 0 & \dots & 0 \\ b_4 - b_2 & 0 & 1 & 0 & \dots & 0 \\ b_5 - b_2 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} - b_2 & 0 & 0 & 0 & \dots & 1 \\ \sum_{k=3}^{n}(b_k - b_2) & 0 & 0 & 0 & \dots & 0 \end{vmatrix}$$

$$= (-1)^{n-1} \sum_{k=3}^{n} (b_k - b_2).$$

Then, using the expression of each b_j, $j \neq 1$, we obtain

$$\begin{aligned} |Ab_1| &= [\delta v_1^{g_S} - v_2^{g_S} + 1 + (n-1)\delta]/2 - \delta \sum_{k=3}^{n} (v_2^{g_S} - v_k^{g_S})/2 \\ &= \frac{1}{2} \left(\delta v_1^{g_S} - [1 + (n-2)\delta] v_2^{g_S} + \delta \sum_{k=3}^{n} v_k^{g_S} + [1 + (n-1)\delta] \right). \end{aligned}$$

Application of Cramer's rule gives us

$$q_{21}^{**} = |Ab_1| \,/\, |A| = \frac{1}{2} + \frac{\delta v_1^{g_S} - [1 + (n-2)\delta] v_2^{g_S} + \delta \sum_{k=3}^{n} v_k^{g_S}}{2[1 + (n-1)\delta]}.$$

To compute any relative price q_{j1}^{**}, $j \neq 1$, one must follow an argument analogous to the one above. Hence, there is a unique SPE s^* for the game $\Gamma(g_S, M)$ such that any peripheral agent $j \neq 1$ charges a relative price

$$q_{j1}^{**} = \frac{1}{2} + \frac{\delta v_1^{g_S} - [1 + (n-2)\delta] v_j^{g_S} + \delta \sum_{k \neq 1, j} v_k^{g_S}}{2[1 + (n-1)\delta]}$$

to the central agent (agent 1) in the star network.

Proof of Proposition 2

We proceed by induction to obtain $|A_{n-1}|$. First, note that

$$A_2 = \begin{vmatrix} 1+\delta & 1 \\ \delta & 1+\delta \end{vmatrix} = 1 + \delta + \delta^2 \quad \text{and} \quad A_3 = \begin{vmatrix} 1+\delta & 1 & 0 \\ \delta & 1+\delta & 1 \\ 0 & \delta & 1+\delta \end{vmatrix} = 1 + \delta + \delta^2 + \delta^3.$$

Also, it can be verified that, for each $m \geq 3$,

$$|A_{m+1}| = (1 + \delta) \, |A_m| - \delta \, |A_{m-1}| \, .$$

Therefore, by proposing $|A_m| = \sum_{j=0}^{m} \delta^j$, we obtain

$$|A_{m+1}| = (1+\delta)\sum_{j=0}^{m} \delta^j - \delta \sum_{j=0}^{m-1} \delta^j = \sum_{j=0}^{m} \delta^j + \delta \left(\sum_{j=0}^{m} \delta^j - \sum_{j=0}^{m-1} \delta^j \right)$$
$$= \sum_{j=0}^{m} \delta^j + \delta\delta^m = \sum_{j=0}^{m+1} \delta^j.$$

It follows that $|A_{n-1}| = \sum_{j=0}^{n-1} \delta^j$. Thus, the linear system $A_{n-1} \cdot q^* = b$ above has a unique solution. We apply Cramer's rule to solve this system.

To compute each solution $q^*_{i(i+1)}$, $i = 1, \ldots, n-1$, we replace the ith column of matrix A_{n-1} with vector b. Let us denote the matrix obtained in this way by Ab_i. To compute the determinant $|Ab_i|$ we use the cofactor expansion along the ith column of matrix Ab_i. It can be checked that

$$|Ab_i| = \frac{\sum_{j=0}^{n-1} \delta^j - \sum_{k=1}^{i} v_k^{g_{\mathrm{L}}} \sum_{j=i}^{n-1} \delta^j + \sum_{k=i+1}^{n} v_k^{g_{\mathrm{L}}} \sum_{j=0}^{i-1} \delta^j}{2}.$$

Then, application of Cramer's rule gives us

$$q^*_{i(i+1)} = |Ab_i| \,/\, |A_{n-1}| = \frac{1}{2} + \frac{\sum_{k=i+1}^{n} v_k^{g_{\mathrm{L}}} \sum_{j=0}^{i-1} \delta^j - \sum_{k=1}^{i} v_k^{g_{\mathrm{L}}} \sum_{j=i}^{n-1} \delta^j}{2\sum_{j=0}^{n-1} \delta^j}.$$

Hence, there is a unique SPE s^* for the game $\Gamma(g_{\mathrm{L}}, M)$ such that each agent $i = 1, \ldots, n-1$ along the line charges the relative price $q^*_{i(i+1)}$ above to agent $i+1$.