NOTES ON A CONSTRAINED SUBOPTIMALITY RESULT BY J.D. GEANAKOPLOS AND H.M. POLEMARCHAKIS (1986)*

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A B S T R A C T

J.D. Geanakoplos and H.M. Polemarchakis (1986) prove the generic constrained suboptimality of equilibrium in two period economies with incomplete markets. In these notes we provide a complete and detailed version of their proof.

1. INTRODUCTION

J. D. Geanakoplos and H. M. Polemarchakis (1986) (GP from here onwards) studied the optimality properties of a two period economy and proved an extremely important result, namely, they showed that when real assets are traded in economies with two or more goods, and markets are incomplete, then the equilibrium allocation is inefficient in the strong sense of being constrained suboptimal, that is, even if the *social planner* is restricted to using the existing assets to obtain the reallocation, he is able to induce an improvement over the equilibrium allocation. That result has become a cornerstone for subsequent research in the area.

The original proof by GP, though complete and correct, skips many details in order to shorten the presentation. In our opinion, understanding the result requires one to have the relevant details of the various arguments. In these notes we provide the said details. We do not provide alternative or new proofs; we simply complete the arguments following the sketches given by GP. So, ours is a purely pedagogical endeavour that we believe permits the reader to appreciate better the nature of the contribution of GP.

The key feature of the proof is to show that, with incomplete markets, the individuals' rates of marginal utilities of income across states differ generically, a fact which is used to show that, if there are two or more commodities, then a relative price effect can be induced in such a way as to cause a welfare improvement. However, to achieve that, one has also to show that a property of linear independence is generically satisfied for a set of vectors derived from the income effect vectors. This property is due to a fact that is independent of the incompleteness of markets, and, to guarantee that it holds, an upper bound needs to be imposed on the number of agents.

GP obtain their result for a generic set of economies where utilities and endowments are used as parameters. Geanakoplos, Magill, Quinzii and Dreze (1990) prove an analogous result for an economy with production, a fact that permits them to consider a generic choice of both producers and consumers endowments, but not of utility functions. Recently, Citanna, Kajii and Villanacci (1998) prove the same result as GP for a pure exchange economy. Their description of the planner's intervention differs from the one used by GP in that (i) individuals are allowed to retrade the assets allocated at the intervention, and (ii) the planner makes lump-sum transfers in some goods. Feature (ii) permits Citanna et. al. to achieve the result without imposing an upper bound on the number of agents.

The rest of the paper is structured as follows. Section 2 presents the model and notation. In Section 3 we introduce the tools that permit us to analyze the effects of the intervention on utilities. In section 4 we obtain two linear independence results derived from the description of the economy. Section 5 deals with the individuals' marginal utilities of income when markets are incomplete. Section 6 presents a technical result on linear algebra. Section 7 explains the functioning of the proof by using the arguments presented, and Section 8 concludes.

2. THE MODEL

Consider a multigood, two period, pure exchange economy under uncertainty described by the realization from a finite set of states of the world.

Let $\mathcal{S} = \{0, 1, ..., s, ..., S\}$, where $S + 1 := \#\mathcal{S}$, be the set of states. There is a set $\mathcal{H} = \{0, 1, ..., h, ..., H\}$ of two period lived agents who care about consumption and reallocate their income intertemporally by trading real assets, defined below, before the state of nature is realized. Assets are traded in the first period and consumption takes place only in the second period. $\mathcal{L} = \{0, 1, ..., l, ..., L\}$ is the set of commodities and $\mathcal{A} = \{0, 1, ..., a, ..., A\}$ is the set of real assets available in the economy. We set $H + 1 := \#\mathcal{H}$, $L + 1 := \#\mathcal{L}$, and $A + 1 := \#\mathcal{A}$.

Consumption of commodity l by individual h in state s is denoted by the non negative number $x_l^h(s)$, $x^h(s) := \left(x_l^h(s)\right)_{l\in\mathcal{L}} \in \Re_+^{L+1}$ indicates individual h's consumption in state s, and $x^h := \left(x^h(s)\right)_{s\in\mathcal{S}} \in \Re_+^{(L+1)(S+1)}$ stands for individual h's consumption plan.[1] Also, let us define an allocation, $x := (x^h)_{h\in\mathcal{H}} \in \Re_+^{(L+1)(S+1)(H+1)}$.

Individual h's preferences are represented by a utility function $u^h : \Re_+^{(L+1)(S+1)} \to \Re$.

Agents' endowments complete the formal description of the characteristics of the economy. As in the case of the consumption variables, the numbers and the vectors $\omega_l^h(s) \in \Re_+$, $\omega^h(s) \in \Re_+^{L+1}$, $\omega^h \in \Re_+^{(L+1)(S+1)}$ and $\omega \in \Re_+^{(L+1)(S+1)(H+1)}$ are used as notation for endowments.

The $A+1$ one period lived *inside real assets* pay a return in terms of commodity 0 in every state $s \in \mathcal{S}$ denoted, for the corresponding security $a \in \mathcal{A}$, by $r_a(s) \in \Re$. Let $r(s) := \left(r_a(s)\right)_{a\in\mathcal{A}} \in \Re^{A+1}$ be the vector of asset returns in state s,
$r_a := \left(r_a(s)\right)_{s\in\mathcal{S}} \in \Re^{S+1}$ be the vector of payoffs of asset a, and

$$R := \begin{bmatrix} r(0)^T \\ r(1)^T \\ \vdots \\ r(S)^T \end{bmatrix} \equiv [r_0 \;\; r_1 \;\; \cdots \;\; r_A]$$

be the corresponding matrix, of dimension $(S+1) \times (A+1)$, of returns.

The quantity of asset $a \in \mathcal{A}$ held by agent $h \in \mathcal{H}$ is denoted by $\theta_a^h \in \Re$, and $\theta^h := (\theta_a^h)_{a\in\mathcal{A}} \in \Re^{A+1}$ denotes individual h's portfolio. We also define an allocation of assets, $\theta := (\theta^h)_{h\in\mathcal{H}} \in \Re^{(A+1)(H+1)}$.

The following standard conditions are assumed to be satisfied by the agents' preferences, by the endowments, and by the asset structure:

ASSUMPTION (A):
(i) For all $h \in \mathcal{H}$, $\omega^h \in \Re_{++}^{(L+1)(S+1)}$,

[1] By convention, for any vector $y \in \Re^{(L+1)(S+1)}$, $y \equiv \Big(y_0(0), ..., y_L(0), ..., y_0(S), ..., y_L(S)\Big)$.

u^h is C^2, strictly monotone, differentiably strictly quasi-concave, and the closure of the indifference curves do not intersect the boundary of $\Re_+^{(L+1)(S+1)}$.

(iia) R has full column rank.

(iib) There exists a portfolio $\theta \in \Re^{A+1}$ such that $R \cdot \theta > \underline{0}$.[2]

(iic) $A < S$.

(iii) Every set of $A+1$ rows of R is linearly independent and there exists a portfolio $\theta \in \Re^{A+1}$ such that $r(s) \cdot \theta \neq 0$ for all $s \in \mathcal{S}$.

We allow for *free disposal* of commodities, and denote the vector of commodity prices by $p := \Big(p(s)\Big)_{s \in \mathcal{S}} \in \Re_+^{(L+1)(S+1)} \setminus \{\underline{0}\}$, where $p(s) := \Big(p_0(s), ..., p_L(s)\Big) \in \Re_+^{L+1} \setminus \{\underline{0}\}$, and the non negative number $p_l(s)$ is the price of commodity l in state s. Also, let $q := (q_0, ..., q_A) \in \Re^{A+1}$ be the vector of asset prices, where q_a is the price of asset a.

Given the nature of the problem, is easy to see that the price of commodity 0 can be chosen as *numeraire* and normalized to 1 in every state $s \in \mathcal{S}$, and the asset prices can be normalized by setting $q_0 = 1$. Normalized prices are denoted with the label ˆ. Moreover, by Walras' law, it suffices to consider markets for just L commodites in every state, and A assets. Commodity 0 and asset 0 correspond to the *dropped* markets. We use the label ˆ for the truncated vectors, where the numeraire commodity and asset are dropped from the vectors.

Now we can define equilibrium.

DEFINITION 1 (CE): $(x^*, \theta^*, \hat{p}^*, \hat{q}^*)$ is a *Competitive Equilibrium (CE)* if:

(i) (a) $\sum_{h \in \mathcal{H}} (\hat{x}^{h^*} - \hat{\omega}^h) \leq \underline{0}$,

(b) $\sum_{h \in \mathcal{H}} \hat{\theta}^{h^*} = \underline{0}$.

(ii) for every $h \in \mathcal{H}$,

(a) $\sum_{a \in \mathcal{A}} \hat{q}_a^* \cdot \theta_a^{h^*} \leq 0$,

$\sum_{l \in \mathcal{L}} \hat{p}_l^*(s) \cdot [x_l^{h^*}(s) - \omega_l^h(s)] \leq \sum_{a \in \mathcal{A}} r_a(s) \cdot \theta_a^{h^*}$ for all $s \in \mathcal{S}$;

(b) if $u^h(x^h) > u^h(x^{h^*})$, for some θ^h, then either

$\sum_{a \in \mathcal{A}} \hat{q}_a^* \cdot \theta_a^h > 0$ or

$\sum_{l \in \mathcal{L}} \hat{p}_l^*(s) \cdot [x_l^h(s) - \omega_l^h(s)] > \sum_{a \in \mathcal{A}} r_a(s) \cdot \theta_a^h$ for some $s \in \mathcal{S}$.

DEFINITION 2 (SM-CE): Given an allocation of assets $\bar{\theta} \in \Re^{(A+1)(H+1)}$ such that $\sum_{h \in \mathcal{H}} \bar{\theta}^h = \underline{0}$, $(x^{**}, \hat{p}^{**})$ is a *Spot Market Competitive Equilibrium (SM-CE)* if:

(i) $\sum_{h \in \mathcal{H}} (\hat{x}^{h^{**}} - \hat{\omega}^h) \leq \underline{0}$.

(ii) for every $h \in \mathcal{H}$,

(a) $\sum_{l \in \mathcal{L}} \hat{p}_l^{**}(s) \cdot [x_l^{h^{**}}(s) - \omega_l^h(s)] \leq \sum_{a \in \mathcal{A}} r_a(s) \cdot \bar{\theta}_a^h$ for all $s \in \mathcal{S}$;

(b) if $u^h(x^h) > u^h(x^{h^{**}})$, then

$\sum_{l \in \mathcal{L}} \hat{p}_l^{**}(s) \cdot [x_l^h(s) - \omega_l^h(s)] > \sum_{a \in \mathcal{A}} r_a(s) \cdot \bar{\theta}_a^h$ for some $s \in \mathcal{S}$.

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[2] When comparing two vectors x and y of the same dimension we use the symbols "$<$", and "$\leq$" to indicate $x_n \leq y_n$ for all n but $x \neq y$, and $x_n \leq y_n$ for all n respectively.

The notion of optimality used is the benchmark in the case where markets are incomplete. It applies the concept of Pareto efficiency to the economy above, but imposing the restriction that any alternative allocation be traded in the existing markets. This yields the criterion of Constrained Pareto Optimality, due to Stiglitz (1982), and Newbery and Stiglitz (1982).

DEFINITION 3 (CS): An allocation (x, θ) is *Constrained Suboptimal (CS)* if there exists an alternative allocation $(\tilde{x}, \tilde{\theta})$, and a price vector $\hat{p} \in \Re_{+}^{(L+1)(S+1)} \setminus \{\underline{0}\}$ such that:
(i) $(\tilde{x}, \hat{p})$ is a SM-CE for the asset allocation $\tilde{\theta}$,
(ii) (a) $u^h(\tilde{x}^h) \geq u^h(x^h)$ for every $h \in \mathcal{H}$;
(b) $u^{h'}(\tilde{x}^{h'}) > u^{h'}(x^{h'})$ for some $h' \in \mathcal{H}$.

Since we will obtain a generic result, we have to work with a set of economies rather than with only one. Such a set is obtained via a parameterization of the economy based on both fundamentals, utilities and endowments. We denote the space of endowments by $\Omega \subset \Re_{++}^{(L+1)(S+1)(H+1)}$, with the requirement that $\omega_l^h(s)$ is bounded away from zero for every $\omega \in \Omega$. Also, consider the set $U := \{f : \Re_{+}^{(L+1)(S+1)} \to \Re$ s.t.: f satisfies Assumption (A): (i) $\}$, and denote the space of utilities by $\mathcal{U} = U \times ... \times U$, (where the product is done $H+1$ times). An element of $\mathcal{U}$ is a list of utility functions, $u = (u^0, ..., u^H)$. The space of economies considered is $\Gamma := \Omega \times \mathcal{U}$.

We can now state the GP result; a detailed development of the proof is the subject matter of the rest of the paper. We remark that, by assuming that u^h has an additively separable representation, part of the proof is made easier.

THEOREM (T): *Assume (A), and that the following holds:*

For every agent $h \in \mathcal{H}$, there is a Bernoulli function $v^h : \Re_{+}^{L+1} \to \Re$, and a probability distribution $\left(\pi^h(s)\right)_{s\in\mathcal{S}} \in \Re_{+}^{S+1}$, such that $u^h(x^h) := \sum_{s\in\mathcal{S}} \pi^h(s) \cdot v^h[x^h(s)]$ for every consumption plan x^h.

Then, given $0 < 2L \leq H < LS$, and $A \geq 1$, there exists a generic set $\tilde{\Gamma} \subset \Gamma$ such that, for all economies $(\omega, u) \in \tilde{\Gamma}$, every CE is CS.

3. PRELIMINARIES

The objective of this section is to present the problem as one of intervention by a *central planner* and to introduce the tools which will allow us to interpret its effects on the individuals' welfare. As a first step, we present two results on the generic regularity of the set of economies described.

We set some notation. Denote by P and Q the vector spaces in which normalized commodity and asset prices, respectively, lie. Given an economy $(\omega, u) \in \Gamma$, and prices $(\hat{p}, \hat{q}) \in P \times Q$, we define the non-numeraire excess demand for commodities and assets $(\hat{z}, \hat{\Theta}) : \Gamma \times P \times Q \to \Re^{L(S+1)} \times \Re^A$,

so that $\hat{z} := \sum_{h\in\mathcal{H}}(\hat{x}^h - \hat{\omega}^h)$, and $\hat{\Theta} := \sum_{h\in\mathcal{H}}\hat{\theta}^h$.
When we fix a specific value for the parameters and consider the resulting non-numeraire aggregate excess demand, we write those parameters as a subscript, e.g., $(\hat{z},\hat{\Theta})_{\omega,u}$ reflects the excess demand function:
$(\hat{z},\hat{\Theta})_{\omega,u} : P \times Q \to \Re^{L(S+1)} \times \Re^A$,
for the specific economy $(\omega, u) \in \Gamma$, and, similarly,
$(\hat{z},\hat{\Theta})_u : \Omega \times P \times Q \to \Re^{L(S+1)} \times \Re^A$
stands for the excess demand of the economy with a fixed utility parameter $u \in \mathcal{U}$ when the endowment $\omega \in \Omega$ is allowed to vary.

A similar notational convention is used for any function parameterized by the fundamentals of the economy.

PROPOSITION 1 (GENERIC REGULARITY): *Given (A): (i), (iia), (iib), given $\bar{u} \in \mathcal{U}$, there exists a generic set $\rho(\bar{u}) \subset \Omega$ such that, for all $\omega \in \rho(\bar{u})$, the set of competitive equilibria is a continuously differentiable function of ω.*

PROOF: Fix a utility parameter $\bar{u} \in \mathcal{U}$ and consider the excess demand function of the non-numeraire commodities and assets:
$(\hat{z},\hat{\Theta})_{\bar{u}} : \Omega \times P \times Q \to \Re^{L(S+1)} \times \Re^A$. Notice that $(\hat{z},\hat{\Theta})^{-1}_{\bar{u}}(\underline{0})$ is the graph of the equilibrium correspondence.

Pick a vector $(\omega', \hat{p}', \hat{q}') \in (\hat{z},\hat{\Theta})^{-1}_{\bar{u}}(\underline{0})$. We perturb endowments in two different ways in order to show that the matrix $D_{(\omega,\hat{p},\hat{q})}(\hat{z},\hat{\Theta})_{\bar{u}}$, evaluated at $(\omega', \hat{p}', \hat{q}')$, has full row rank.

First, consider the following choices:
(a) agent $0 \in \mathcal{H}$,
(b) a given commodity $l' \in \mathcal{L} \setminus \{0\}$,
(c) a given state $s' \in \mathcal{S}$.

Perturb ω^0 by adding $[-\hat{p}_{l'}(s')]$ to the coordinate that corresponds to commodity 0 in state s', and by adding $[+1]$ to the coordinate that corresponds to commodity l' in state s'. The induced change in agent 0's income for the spot market in state s' is

$$1 \cdot [-\hat{p}_{l'}(s')] + \hat{p}_{l'}(s') \cdot [+1] = 0,$$

where the first element of these products reflects prices and the second stands for quantities. Hence, by construction of the perturbation, individual 0's optimal choice is unaltered, and so is aggregate demand. However, $\hat{z}$ is additively changed by an amount indicated by a vector of dimension $L(S+1)$ that contains -1 in the coordinate that corresponds to good l' in state s', and zero in the other coordinates.

Consider now the following choices:
(a) agent $0 \in \mathcal{H}$,
(b) a given asset $a' \in \mathcal{A}$.

Perturb ω^0 by adding $[r_{a'}(s) - \hat{q}_{a'} \cdot r_0(s)]$ to the coordinates that correspond to commodity 0 in every state $s \in \mathcal{S}$. Now we need to compensate the induced change in agent 0's income to maintain unaltered the equilibrium prices. This can be done by adding

$[+\hat{q}_{a'}]$ to agent 0's holding of asset 0, and by adding $[-1]$ to his holding of asset a'. So, the change in income in s is

$$1 \cdot [r_{a'}(s) - \hat{q}_{a'} \cdot r_0(s)] + 1 \cdot [r_0(s) \cdot \hat{q}_{a'} - r_{a'}(s) \cdot 1] = 0.$$

Since $\sum_{a \in \mathcal{A}} r_a(s) \cdot \theta_a^0 \leq 0$, the perturbed asset holding continues to be budget feasible. It follows that individual 0's optimal choice is unaltered.

At the prices $(\hat{p}', \hat{q}')$, the perturbation alters $\hat{z}$ by the addition of a vector of dimension $L(S+1)$ that contains -1 in the coordinate that corresponds to good l' in state s', and zero in the other coordinates, and $\hat{\Theta}$ by the addition of a vector that contains -1 in the coordinate that corresponds to asset a' and zero in the others. Hence, the matrix $D_{(\omega,\hat{p},\hat{q})}(\hat{z}, \hat{\Theta})_{\bar{u}}$, evaluated at a vector $(\omega', \hat{p}', \hat{q}') \in (\hat{z}, \hat{\Theta})_{\bar{u}}^{-1}(\underline{0})$, has full row rank.

Now, by applying a transversality argument, we know that there exists a generic set $\rho(\bar{u}) \subset \Omega$ such that $D_{(\hat{p},\hat{q})}(\hat{z}, \hat{\Theta})_{\omega,\bar{u}}$ has full row rank when evaluated at prices $(\hat{p}', \hat{q}') \in (\hat{z}, \hat{\Theta})_{\omega,\bar{u}}^{-1}(\underline{0})$, where $\omega \in \rho(\bar{u})$ and $(\hat{z}, \hat{\Theta})_{\omega,\bar{u}}^{-1}(\underline{0})$ is the graph of the associated equilibrium correspondence. The result follows by applying the Implicit Function Theorem.

■

Notice that the parameters perturbed up to now are those reflected by the vector ω_0^0, and the number $\omega_{l'}^0(s')$ for every $l' \in \mathcal{L} \setminus \{0\}$ and for every $s' \in \mathcal{S}$.

We specify the notation for the generic set of economies identified in Proposition 1 as $\Gamma_1 := \{(u, \omega) \in \Gamma : u \in \mathcal{U}, \omega \in \rho(u)\}$.

Since, by Proposition 1, equilibria are isolated, utility functions can be perturbed by the addition of a quadratic term in a way such that the linear term subsequently added to the vector of the first derivatives amounts to zero at the equilibrium allocation. Therefore, the perturbation leaves unaffected demand but it changes the matrix of second derivatives of the utility function. Using this fact, it can be shown that any perturbation of the matrix $D_{\hat{p}}\hat{z}^h$, $h \in \mathcal{H}$, by the addition of a symmetric matrix can be induced by adding a suitably chosen quadratic term to the utility function of such agent.[3] This result will be used in the next proposition.

PROPOSITION 2 (GENERIC STRONG REGULARITY): *Given (A): (i), (iia), (iib), given a feasible asset allocation, $\bar{\theta} \in \Re^{(A+1)(H+1)}$, and the corresponding vector of equilibrium commodity prices, $\bar{\hat{p}} \in \Re_+^{L(S+1)}$, the matrix $D_{\hat{p}}\hat{z}$ is invertible when evaluated at $\bar{\hat{p}}$, for a generic set of economies $\Gamma_2 \subset \Gamma_1$.*

PROOF: Consider the non-numeraire aggregate excess demand function $(\hat{z}, \hat{\Theta}) : \Gamma \times P \times Q \to \Re^{L(S+1)} \times \Re^A$. Denote the non-numeraire excess demand for commodities of individual $h \in \mathcal{H}$ at the portfolio and prices $(\theta^h, \hat{p}) \in \Re^{A+1} \times \Re_+^{L(S+1)}$ by $\hat{z}^h(\theta^h, \hat{p}) := (\hat{x}^h - \hat{\omega}^h)$, where $\hat{x}^h$ solves individual h's decision problem given $(\theta^h, \hat{p})$. Also, denote by $\hat{z}(\theta, \hat{p}) := \sum_{h \in \mathcal{H}} \hat{z}^h(\theta^h, \hat{p})$ the corresponding non-numeraire aggregate excess demand.

[3] See, for example, Geanakoplos and Polemarchakis (1980).

Define the function:

$G : \Gamma \times P \times Q \times \Delta^{L(S+1)-1} \rightarrow \Re^{L(S+1)} \times \Re^{A} \times \Re^{L(S+1)}$,

where $\Delta^{L(S+1)-1}$ is the simplex of dimension $L(S+1)-1$, and G is specified as $G(\omega, u, \hat{p}, \hat{q}, \delta) := [(\hat{z}, \hat{\Theta})(\omega, u, \hat{p}, \hat{q}), \delta^T \cdot D_{\hat{p}}\hat{z}(\bar{\theta}, \hat{p})]$.

Consider an array $(\omega, u, \bar{\hat{p}}, \bar{\hat{q}}) \in (\hat{z}, \hat{\Theta})^{-1}(\underline{0})$. By using the result obtained in Proposition 1, and by applying a transversality argument, we know that $(\hat{z}, \hat{\Theta})_{\omega,u}$ is generically transverse to zero. Now, given a set of matrices that admit symmetric perturbations, we can extract a generic subset all of whose elements are invertible matrices. This, together with the fact noted above that, for any individual $h \in \mathcal{H}$, it is possible to perturb D^2u^h, maintaining Du^h at its original value, changing $D_{\hat{p}}\hat{z}^h$ by the addition of a symmetric matrix, leads to the result that $G_{\omega,u}$ is generically transverse to zero. By applying the Regular Value Theorem, we know that $G^{-1}_{\omega,u}(\underline{0}) = \emptyset$ for a generic set of economies $\Gamma_2 \subset \Gamma_1$, since the dimension of the domain of $G_{\omega,u}$ is less than the dimension of the range. Hence, there does not exist any $\delta \in \Delta^{L(S+1)-1}$ such that $\delta^T \cdot D_{\hat{p}}\hat{z} = \underline{0}^T$, and, consequently, $D_{\hat{p}}\hat{z}$ is generically invertible when evaluated at $\bar{\hat{p}}$.

∎

The parameters altered to obtain the regularity results above consist of ω_0^0, $\omega_{l'}^0(s')$ for every $l' \in \mathcal{L} \setminus \{0\}$ and for every $s' \in \mathcal{S}$, and u^h for some $h \in \mathcal{H}$.

Let us now introduce a *central planner*, who reallocates the existing assets among the individuals before trade takes place. After that intervention, agents are allowed to trade in the markets for goods to the point where a new equilibrium in the commodity markets is reached. However, they are not allowed to retrade the asset portfolio they were assigned. We wish to show that there exists an asset reallocation such that the induced allocation of commodities is Pareto improving for at least a generic set of economies.

The asset redistribution directly affects the agents' income and, given that more than a single good is traded, also affects relative commodity prices in the spot markets at the next date. Both types of effects change the individuals' budget sets and therefore their consumption possibilities. However, intuitively we can see that the direct effect of any feasible asset reallocation on the individuals' income does not permit a Pareto improvement, since only a redistribution of a fixed amount of income takes place among the agents. Therefore, the key to analysing the final effects on welfare rests in proposing a reallocation of asset holdings such that, taking into account the induced price effect, the *new* equilibrium allocation, which will be induced by the *new* asset holdings, is a constrained Pareto improvement.

We proceed by explicitly setting up the optimization problem of an agent $h \in \mathcal{H}$:

$\max_{\{x^h, \theta^h\}} u^h(x^h)$

subject to: $\sum_{a \in \mathcal{A}} \hat{q}_a \cdot \theta_a^h \leq 0$

$$\sum_{l\in\mathcal{L}} \hat{p}_l(s) \cdot [x_l^h(s) - \omega_l^h(s)] \leq \sum_{a\in\mathcal{A}} r_a(s) \cdot \theta_a^h \quad \text{for all } s \in \mathcal{S}.$$

Those budget constraints hold with equality at the solution, given Assumption (A): (i). The first order conditions for an interior optimal choice are:

$$\hat{q}_a \cdot \mu^h = \sum_{s\in\mathcal{S}} r_a(s) \cdot \lambda^h(s) \quad \text{for all } a \in \mathcal{A}, \text{ and}$$
$$\frac{\partial u^h(x^h)}{\partial x_l^h(s)} = \lambda^h(s) \cdot \hat{p}_l(s) \quad \text{for all } l \in \mathcal{L},\ s \in \mathcal{S},$$

where μ^h, $\lambda^h(s)$ are, respectively, the Lagrange multipliers for individual h's budget constraints on assets and for the spot market in state s.

Consider the change induced by perturbing individual h's asset holding on his consumption plan, where we allow prices to change. From the first order conditions, and noting that $du^h = \left[D_{x^h} u^h\right]^T \cdot dx^h$, the change in utility of agent h at the margin due to a marginal change in his consumption plan is

$$du^h = \sum_{s\in\mathcal{S}} \lambda^h(s) \cdot \sum_{l\in\mathcal{L}} \hat{p}_l(s) \cdot dx_l^h(s). \tag{1}$$

Now we can consider the changes induced by such an asset perturbation on the individuals' consumption plans. By totally differentiating individual h's spot market budget constraint, we obtain:

$$\sum_{l\in\mathcal{L}} d\hat{p}_l(s) \cdot [x_l^h(s) - \omega_l^h(s)] + \sum_{l\in\mathcal{L}} \hat{p}_l(s) \cdot dx_l^h(s) - \sum_{a\in\mathcal{A}} r_a(s) \cdot d\theta_a^h = 0,$$

for each $s \in \mathcal{S}$, a condition that must be satisfied by the induced changes at any equilibrium. Equivalently,

$$\sum_{l\in\mathcal{L}} \hat{p}_l(s) \cdot dx_l^h(s) = \sum_{a\in\mathcal{A}} r_a(s) \cdot d\theta_a^h - \sum_{l\in\mathcal{L}} d\hat{p}_l(s) \cdot [x_l^h(s) - \omega_l^h(s)] = \sum_{a\in\mathcal{A}} r_a(s) \cdot d\theta_a^h - \sum_{l\in\mathcal{L}\setminus\{0\}} d\hat{p}_l(s) \cdot \hat{z}_l^h(s), \tag{2}$$

where we use the fact that the price of the numeraire commodity does not change, and where $\hat{z}_l^h(s) := [x_l^h(s) - \omega_l^h(s)]$, for all $l \in \mathcal{L} \setminus \{0\}$.

By combining equations (1) and (2), we obtain:

$$du^h = \sum_{s\in\mathcal{S}} \lambda^h(s) \cdot \left[\sum_{a\in\mathcal{A}} r_a(s) \cdot d\theta_a^h - \sum_{l\in\mathcal{L}\setminus\{0\}} d\hat{p}_l(s) \cdot \hat{z}_l^h(s)\right], \tag{3}$$

where the first element in brackets reflects the direct effect of the asset reallocation on individual h's utility through a perturbation of the agent's income, and the second reflects the contribution due to the change in relative prices. We turn now to a more detailed analysis of this price effect.

Let us recall that, after the reallocation of assets takes place, the markets for goods open again and they clear at some *new* equilibrium prices. Considering the non-numeraire aggregate excess demand as a function of the commodity prices and the asset allocation, we see that, at the original equilibrium, $\hat{z}(\theta, \hat{p}) = \underline{0}$, and that, by totally differentiating this equality, $D_{\hat{p}}\hat{z}d\hat{p} + D_\theta \hat{z} d\theta = \underline{0}$. By invoking the Strong Regularity result, Proposition 2, we know that, for the generic set of economies Γ_2, $D_{\hat{p}}\hat{z}$ is invertible when evaluated at an equilibrium price vector. So, we can apply the Implicit Function Theorem to say that

$$d\hat{p} = -(D_{\hat{p}}\hat{z})^{-1} \cdot D_\theta \hat{z} \cdot d\theta \tag{4}$$

holds in a neighbourhood of the original equilibrium. Our problem has been reduced to specifying an asset perturbation such that an improvement can be induced, where the change in utility of agent h is given by (3), and the change in prices is determined by the matrix $D_\theta \hat{z}$, of dimension $L(S+1) \times (H+1)(A+1)$, that appears in equation (4). We turn to a detailed analysis of that matrix.

Consider an asset reallocation in which individual 0 makes a gift of asset 0 to all agents $h \in \mathcal{H} \setminus \{0\}$ and gifts asset 1 to agent 1.[4]

The set of perturbations that we consider is $\{\widetilde{d\theta_0^1}, ..., \widetilde{d\theta_0^H}, \widetilde{d\theta_1^1}\}$, where $\widetilde{d\theta_{a'}^{h'}}$ denotes the gift of asset $a' \in \{0,1\}$ obtained by agent $h' \in \mathcal{H} \setminus \{0\}$ from individual 0. Notice that the overall feasibility requirements allow us to express the variation which takes place in individual 0's asset holdings by $\widetilde{d\theta_0^0} = -\sum_{h' \in \mathcal{H} \setminus \{0\}} \widetilde{d\theta_0^{h'}}$, and $\widetilde{d\theta_1^0} = -\widetilde{d\theta_1^1}$. From the proposed set of asset reallocations we specify the vector

$$\widetilde{d\theta} := \begin{pmatrix} \widetilde{d\theta_0^1} \\ \vdots \\ \widetilde{d\theta_0^H} \\ \widetilde{d\theta_1^1} \end{pmatrix},$$

of dimension $H+1$, which reflects the *central planner's* intervention. Notice that the feasibility of the proposed policy is maintained when altering independently any coordinate of the vector above. Therefore, $H+1$ independent instruments are available to the planner by construction; as will be explained, that number is sufficient to generate the *required* changes in the $H+1$ agents' budget sets leading to a Pareto improvement of the allocation at the *new* equilibrium, for a generic set of economies.

Let us step back for a moment and consider a general reallocation of assets. In general, $d\theta^h \in \Re^{A+1}$ for every $h \in \mathcal{H}$, and

$$d\theta = \begin{pmatrix} d\theta^0 \\ d\theta^1 \\ \vdots \\ d\theta^H \end{pmatrix} \in \Re^{(H+1)(A+1)}$$

denotes an asset reallocation; feasibility of the reallocation needs to be imposed separately. The specific perturbation that interests us, which we now specify, will be denoted by $\check{d\theta} := (\check{d\theta^h})_{h \in \mathcal{H}} \in \Re^{(H+1)(A+1)}$, and is obtained as

[4] GP consider a transfer of asset 1 from agent 0 to all agents $h \in \mathcal{H} \setminus \{0\}$, and of asset 0 from individual 0 to 1. Clearly, this variation with respect to what GP propose is of no material consequence.

$$d\check{\theta}^0 := \begin{pmatrix} -\sum_{h'\in\mathcal{H}\setminus\{0\}} \widetilde{d\theta_0^{h'}} \\ -\widetilde{d\theta_1^1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; d\check{\theta}^1 := \begin{pmatrix} \widetilde{d\theta_0^1} \\ \widetilde{d\theta_1^1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ and } d\check{\theta}^h := \begin{pmatrix} \widetilde{d\theta_0^h} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ for every}$$

$h \in \mathcal{H} \setminus \{0, 1\}$.

Evidently, even though $d\check{\theta}$ and $\widetilde{d\theta}$ are vectors of different dimension, they denote the same economic object; however, $d\check{\theta}$ contains zeros in all the coordinates except in those corresponding to the change of asset 1 for individuals 0 and 1, and of asset 0 for all individuals. Consequently, even though the matrix $D_\theta \hat{z}$ has dimension $L(S+1)\times(H+1)(A+1)$, since we are interested in the effect of the specific reallocation on demand, the reader should have in mind that it suffices to consider a submatrix, denoted by $D_{\tilde{\theta}}\hat{z}$, of dimension $L(S+1) \times (H+1)$. The matrix $D_{\tilde{\theta}}\hat{z}$ consists of those entries of matrix $D_\theta \hat{z}$ which are relevant in the sense that the reallocation $d\check{\theta}$ has non-zero entry. From here on, the formalities will be conducted in terms of $D_\theta \hat{z}$ and $d\check{\theta}$, but the intuition for the specific intervention will be explained in terms of the notation $D_{\tilde{\theta}}\hat{z}$ and $\widetilde{d\theta}$.

Let us consider in more detail the income effects generated by the asset intervention. By assuming that u^h has an additively separable representation, we know that a variation of agent h's income in state s only affects his consumption in that state, so that cross influences among states are avoided when working with income effects, a fact that permits us to use the notation that we now introduce for further arguments. For $h \in \mathcal{H}$, the number $V_l^h(s)$ denotes the change of individual h's demand of good l as a consequence of an infinitesimal perturbation of his income in state s. Set the vectors $V^h(s) := \left(V_l^h(s)\right)_{l\in\mathcal{L}\setminus\{0\}}$ and $V^h := \left(V^h(s)\right)_{s\in\mathcal{S}}$, and, for $a \in \mathcal{A}$, define the vector, of dimension $L(S+1)$,

$$V^h \odot r_a := \begin{pmatrix} r_a(0)\cdot V^h(0) \\ r_a(1)\cdot V^h(1) \\ \vdots \\ r_a(S)\cdot V^h(S) \end{pmatrix}.$$

For the proposed policy, by considering the vectors defined above, we can express the changes in the excess demand of the individuals as:

(a) $D_\theta \hat{z}^1 \cdot d\check{\theta} = (V^1 \odot r_0)\cdot \widetilde{d\theta_0^1} + (V^1 \odot r_1)\cdot \widetilde{d\theta_1^1}$,
(b) $D_\theta \hat{z}^{h'} \cdot d\check{\theta} = (V^{h'} \odot r_0)\cdot \widetilde{d\theta_0^{h'}}$ for every $h' \in \mathcal{H}\setminus\{0,1\}$,
(c) $D_\theta \hat{z}^0 \cdot d\check{\theta} = -\sum_{h\in\mathcal{H}\setminus\{0\}}(V^0 \odot r_0)\cdot \widetilde{d\theta_0^h} - (V^0 \odot r_1)\cdot \widetilde{d\theta_1^1}$.

Since $d\hat{z}^h = D_\theta \hat{z}^h \cdot d\check{\theta}$, for every $h \in \mathcal{H}$, $d\hat{z} = D_\theta \hat{z}\cdot d\check{\theta}$, and $d\hat{z} = \sum_{h\in\mathcal{H}} d\hat{z}^h$, we know that

$$D_\theta \hat{z} \cdot \check{d\theta} = \sum_{h \in \mathcal{H} \setminus \{0\}} (V^h - V^0) \odot r_0 \cdot \widetilde{d\theta_0^h} + (V^1 - V^0) \odot r_1 \cdot \widetilde{d\theta_1^1}. \tag{5}$$

Given a vector of utilities $u \in \mathcal{U}$, we are interested in obtaining a precise specification of the matrix of derivatives $D_\theta u$, for the policy proposed above. Consider a vector of utilities $u = (u^h)_{h \in \mathcal{H}} \in \mathcal{U}$. By using equation (3), we know that

$$du = \begin{bmatrix} \sum_{s \in \mathcal{S}} \lambda^0(s) \cdot \sum_{a \in \mathcal{A}} r_a(s) \cdot \check{d\theta}_a^0 \\ \sum_{s \in \mathcal{S}} \lambda^1(s) \cdot \sum_{a \in \mathcal{A}} r_a(s) \cdot \check{d\theta}_a^1 \\ \vdots \\ \sum_{s \in \mathcal{S}} \lambda^H(s) \cdot \sum_{a \in \mathcal{A}} r_a(s) \cdot \check{d\theta}_a^H \end{bmatrix} + \begin{bmatrix} -\sum_{s \in \mathcal{S}} \lambda^0(s) \cdot \sum_{l \in \mathcal{L} \setminus \{0\}} d\hat{p}_l(s) \cdot \hat{z}_l^0(s) \\ -\sum_{s \in \mathcal{S}} \lambda^1(s) \cdot \sum_{l \in \mathcal{L} \setminus \{0\}} d\hat{p}_l(s) \cdot \hat{z}_l^1(s) \\ \vdots \\ -\sum_{s \in \mathcal{S}} \lambda^H(s) \cdot \sum_{l \in \mathcal{L} \setminus \{0\}} d\hat{p}_l(s) \cdot \hat{z}_l^H(s) \end{bmatrix}.$$

Equivalently,

$$du = \begin{bmatrix} \sum_{s \in \mathcal{S}} \lambda^0(s) \cdot \sum_{a \in \mathcal{A}} r_a(s) \cdot \check{d\theta}_a^0 \\ \sum_{s \in \mathcal{S}} \lambda^1(s) \cdot \sum_{a \in \mathcal{A}} r_a(s) \cdot \check{d\theta}_a^1 \\ \vdots \\ \sum_{s \in \mathcal{S}} \lambda^H(s) \cdot \sum_{a \in \mathcal{A}} r_a(s) \cdot \check{d\theta}_a^H \end{bmatrix} + \begin{bmatrix} -\sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L} \setminus \{0\}} \lambda^0(s) \cdot \hat{z}_l^0(s) \cdot d\hat{p}_l(s) \\ -\sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L} \setminus \{0\}} \lambda^1(s) \cdot \hat{z}_l^1(s) \cdot d\hat{p}_l(s) \\ \vdots \\ -\sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L} \setminus \{0\}} \lambda^H(s) \cdot \hat{z}_l^H(s) \cdot d\hat{p}_l(s) \end{bmatrix}. \tag{6}$$

We now introduce some extra notation to express (6) in the form $du = (T + C) \cdot \widetilde{d\theta}$, where T and C are two matrices of dimension $(H+1) \times (H+1)$. Let us define, for $h \in \mathcal{H}$, the row vector, of dimension $L(S+1)$,

$$\lambda^h \odot \hat{z}^h := \left(\lambda^h(0) \cdot \left[\hat{z}^h(0)\right]^T \quad \lambda^h(1) \cdot \left[\hat{z}^h(1)\right]^T \quad \cdots \quad \lambda^h(S) \cdot \left[\hat{z}^h(S)\right]^T \right).$$

Define also the matrix, of dimension $(H+1) \times L(S+1)$,

$$(\widetilde{\lambda \odot \hat{z}}) := \begin{bmatrix} \lambda^0 \odot \hat{z}^0 \\ \lambda^1 \odot \hat{z}^1 \\ \vdots \\ \lambda^h \odot \hat{z}^H \end{bmatrix}.$$

We can write, for every $h \in \mathcal{H}$,

$$\textstyle\sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L} \setminus \{0\}} \lambda^h(s) \cdot \hat{z}_l^h(s) \cdot d\hat{p}_l(s) = (\lambda^h \odot \hat{z}^h) \cdot d\hat{p},$$

and, hence, the second vector in equation (6) can be expressed as $-(\widetilde{\lambda \odot \hat{z}}) \cdot d\hat{p}$.

Now, by considering

$$d\hat{p} = -(D_{\hat{p}} \hat{z})^{-1} \cdot D_\theta \hat{z} \cdot \check{d\theta},$$

obtained earlier, and by using equation (5), we know that

$$\begin{aligned} &-(\widetilde{\lambda \odot \hat{z}}) \cdot d\hat{p} = (\widetilde{\lambda \odot \hat{z}}) \cdot (D_{\hat{p}} \hat{z})^{-1} D_\theta \hat{z} \cdot \check{d\theta} = \\ &(\widetilde{\lambda \odot \hat{z}}) \cdot (D_{\hat{p}} \hat{z})^{-1} \cdot \left(\textstyle\sum_{h \in \mathcal{H} \setminus \{0\}} (V^h - V^0) \odot r_0 \cdot \widetilde{d\theta_0^h} + (V^1 - V^0) \odot r_1 \cdot \widetilde{d\theta_1^1} \right) = \\ &(\widetilde{\lambda \odot \hat{z}}) \cdot (D_{\hat{p}} \hat{z})^{-1} \cdot \left[(V^1 - V^0) \odot r_0 \quad \cdots \quad (V^H - V^0) \odot r_0 \quad (V^1 - V^0) \odot r_1 \right] \cdot \widetilde{d\theta}. \end{aligned}$$

We remark that the matrix

$$\Big[(V^1 - V^0)\odot r_0 \quad \cdots \quad (V^H - V^0)\odot r_0 \quad (V^1 - V^0)\odot r_1\Big]$$

has dimension $L(S+1)\times(H+1)$.

Set the following matrices, of dimension $(H+1)\times(H+1)$,

$C := (\widetilde{\lambda \odot \hat{z}})\cdot(D_{\hat{p}}\hat{z})^{-1}\cdot\Big[(V^1 - V^0)\odot r_0 \quad \cdots \quad (V^H - V^0)\odot r_0 \quad (V^1 - V^0)\odot r_1\Big]$, and

the matrix T that satisfies $T\cdot\widetilde{d\theta} = \begin{bmatrix} \sum_{s\in\mathcal{S}}\lambda^0(s)\cdot\sum_{a\in\mathcal{A}} r_a(s)\cdot d\check{\theta}_a^0 \\ \sum_{s\in\mathcal{S}}\lambda^1(s)\cdot\sum_{a\in\mathcal{A}} r_a(s)\cdot d\check{\theta}_a^1 \\ \vdots \\ \sum_{s\in\mathcal{S}}\lambda^H(s)\cdot\sum_{a\in\mathcal{A}} r_a(s)\cdot d\check{\theta}_a^H \end{bmatrix}$.

Then, from equation (6), we see that du is in fact obtained by considering the sum of these two matrices multiplied by the vector $\widetilde{d\theta}$, that is, $du = (T+C)\cdot\widetilde{d\theta}$.

Notice that if the matrix $T+C$ has rank $H+1$ then it is possible to choose a perturbation $\widetilde{d\theta}$ such that the utility vector changes by the addition of an arbitrary vector; in particular, an improvement can be induced. A standard argument shows that the matrix T cannot have rank $H+1$, since it only captures the effect of a pure redistribution of income. It follows that, to prove theorem (T), it suffices to show that the matrix C specified above has rank $H+1$ for a generic set of economies.

Before we proceed to the details spelled out in the sections to follow, we describe briefly the different arguments that will be used to complete the proof. Our objective is to show that, generically, there is no $\delta\in\Delta^H$ such that $\delta^T\cdot C = \underline{0}^T$. To do that, on the one hand, in Section 4.2, we show that, generically, the matrix obtained by eliminating the vectors of the matrix

$$\Big[(V^1 - V^0)\odot r_0 \quad \cdots \quad (V^H - V^0)\odot r_0 \quad (V^1 - V^0)\odot r_1\Big]$$

that correspond to some dropped state, has rank $H+1$. On the other hand, we show that, given $\delta\in\Delta^H$, by suitably perturbing utilities and endowments, we can alter as we wish the entries of the vector $\delta^T\cdot(\widetilde{\lambda\odot\hat{z}})\cdot(D_{\hat{p}}\hat{z})^{-1}$ that correspond to at least S states, and yet leave $(D_{\hat{p}}\hat{z})^{-1}$ unaffected. To do so we use a result from linear algebra given in Section 6, together with (i) a result on linear independence given in Subsection 4.1, and (ii) Property 2 in Section 5 whereby there exists a set of $L+1$ individuals $\{h_0, h_1, ..., h_L\}\subset\mathcal{H}$, such that, given $\delta = (\delta_h)_{h\in\mathcal{H}}\in\Delta^H$, generically, $0\neq\delta_{h_0}\cdot\lambda^{h_0}(s)\neq\delta_{h_i}\cdot\lambda^{h_i}(s)$ for at least S states, for every $i\in\{1,2...,L\}$.

4. RESULTS ON LINEAR INDEPENDENCE

In this section we obtain two properties of linear independence that the set of vectors $\{V^0, V^1, \cdots, V^H\}$ generically satisfies. These results require that $L>0$ and that preferences not be quasi-linear since otherwise income effects are absent.

4.1. THE SPANNING PROPERTY

The property obtained in this subsection can be stated as follows:

For any subset of $L+1$ individuals, $\{h_0, h_1, \cdots, h_L\} \subset \mathcal{H}$, and for any $s \in \mathcal{S}$, the set of vectors $\{V^{h_1}(s) - V^{h_0}(s), V^{h_2}(s) - V^{h_0}(s), \cdots, V^{h_L}(s) - V^{h_0}(s)\}$ is linearly independent, for a generic set of economies $\Gamma_3 \subset \Gamma$.

PROOF: Consider a subset of $L+1$ individuals arbitrarily drawn from $\mathcal{H}$, $\{h_0, h_1, \cdots, h_L\} \subset \mathcal{H}$, and a state $s \in \mathcal{S}$.

Define the matrix, of dimension $L \times L$,

$$\Pi(s) := \Big[V^{h_1}(s) - V^{h_0}(s) \quad V^{h_2}(s) - V^{h_0}(s) \quad \cdots \quad V^{h_L}(s) - V^{h_0}(s)\Big].$$

Also, define the function

$$\Upsilon(s) : \Gamma \times P \times Q \times \Delta^{L-1} \to \Re^{L(S+1)} \times \Re^A \times \Re^L,$$

where Δ^{L-1} is the simplex of dimension $L-1$, and $\Upsilon(s)$ is specified as $\Upsilon(s)(\omega, u, \hat{p}, \hat{q}, \delta) := [(\hat{z}, \hat{\Theta})(\omega, u, \hat{p}, \hat{q}), \delta^T \cdot \Pi(s)]$.

Since utility functions can be perturbed without changing their first derivatives at the equilibrium allocation, we are able to change $V^h(s)$ for any $h \in \mathcal{H}$ and for any $s \in \mathcal{S}$, maintaining $(\hat{z}, \hat{\Theta})$ unaltered at the equilibrium prices. Therefore, by applying a transversality argument, we know that $\Upsilon(s)_{(\omega,u)}$ is transverse to zero for every $(\omega, u) \in \Gamma_3$, where $\Gamma_3 \subset \Gamma$ is a generic set. Now, given that the dimension of the range of $\Upsilon(s)_{(\omega,u)}$ exceeds that of the domain, by applying the Regular Value Theorem, $\Upsilon(s)^{-1}_{(\omega,u)}(\underline{0}) = \emptyset$ for all $(\omega, u) \in \Gamma_3$, and, hence, we can conclude that $\Upsilon(s)$ has rank L for a generic set of economies Γ_3.

The result follows by noting that s was chosen arbitrarily.

■

Notice that if this property holds then, for any given $s \in \mathcal{S}$, the set of vectors $\{V^{h_1}(s) - V^{h_0}(s), V^{h_2}(s) - V^{h_0}(s), \cdots, V^{h_L}(s) - V^{h_0}(s)\}$ spans $\Re^L$.

4.2. THE CHANGES INDUCED IN THE DEMANDS OF AGENTS BY THE REALLOCATION OF ASSETS

The property that we obtain in this subsection is stated as follows:

The set of vectors $\{(V^1 - V^0) \odot r_0, \cdots, (V^H - V^0) \odot r_0, (V^1 - V^0) \odot r_1\}$ is linearly independent even when considering any LS coordinates of such vectors, for a generic set of economies $\Gamma_4 \subset \Gamma$.

PROOF: Pick a state $s \in \mathcal{S}$ and construct the set of vectors

$$\left\{\overline{(V^1 - V^0) \odot r_0}^s, \cdots, \overline{(V^H - V^0) \odot r_0}^s, \overline{(V^1 - V^0) \odot r_1}^s\right\}$$

by dropping the L coordinates that correspond to state s from the vectors of the set

$$\{(V^1 - V^0) \odot r_0, \cdots, (V^H - V^0) \odot r_0, (V^1 - V^0) \odot r_1\}.$$

We proceed to the proof by decomposing it into two parts.

STEP 1: Consider the vectors $r_0, r_1 \in \Re^{S+1}$. By applying Assumption (A): (iia) and (iii), we know that any pair of vectors of dimension $A+1$ that can be extracted from r_0 and r_1 by dropping from each of them the same coordinates are linearly independent. This implies that r_0 and r_1 are linearly independent when restricted to any number of the coordinates between $A+1$ and $S+1$, and, in particular, when restricted to S of the coordinates.

In addition, Assumption (A): (iii) also guarantees that, without loss of generality, all the components of the vector r_0 can be considered different from zero.

Therefore, the pair of vectors $\overline{(V^1 - V^0) \odot r_0}^s, \overline{(V^1 - V^0) \odot r_1}^s$ are linearly independent, given that, by multiplying $(V^1 - V^0)$ by r_0, and by r_1, according to the product $\odot$, we can see that the vectors r_0 and r_1 are affected by the same proportion in the same coordinates, and, hence, no relative change accross the coordinates is induced.

STEP 2: Define the matrix, of dimension $LS \times (H+1)$,

$$\Sigma^s := \left[\overline{(V^1 - V^0) \odot r_0}^s \quad \cdots \quad \overline{(V^H - V^0) \odot r_0}^s \quad \overline{(V^1 - V^0) \odot r_1}^s \right],$$

and the function

$$\Phi^s : \Gamma \times P \times Q \times \Delta^H \rightarrow \Re^{L(S+1)} \times \Re^A \times \Re^{LS},$$

where
$\Phi^s(\omega, u, \hat{p}, \hat{q}, \delta) := [(\hat{z}, \hat{\Theta})(\omega, u, \hat{p}, \hat{q}), \Sigma^s \cdot \delta]$.

Since we can perturb utility functions in a way such that $(V^h - V^0)$, $h \in \mathcal{H} \setminus \{0\}$, and, consequently, also $\overline{(V^h - V^0) \odot r_0}^s$, $h \in \mathcal{H} \setminus \{0\}$, and $\overline{(V^1 - V^0) \odot r_1}^s$ are changed, maintaining $(\hat{z}, \hat{\Theta})$ unaffected, we know that Φ^s is transverse to zero. Since the dimension of the range of $\Phi^s_{\omega,u}$ exceeds that of the domain, we know that, for all $(\omega, u) \in \Gamma_4$, where $\Gamma_4 \subset \Gamma$ is a generic set, there is no $\delta \in \Delta^H$ such that $\Sigma^s \cdot \delta = \underline{0}$, and, hence, $rank[\Sigma^s] = H+1$.

The result yields since state s was chosen arbitrarily.

■

Notice that, since we are considering a set of $H+1$ vectors, and the linear independence property is stated for at least LS of the coordinates, then $H+1 \leq LS$ appears as a necessary condition for this result to hold. By assuming that $H < LS$, such a condition is satisfied.

We remark that Citanna, Kajii and Villanacci (1998) do not impose an upper bound on the number of agents. They can achieve the constrained suboptimality result since they consider a policy where lump-sum transfers take place for some goods among individuals in the first period. This allows them to control directly the individuals' income effects. They analyze the same problem as GP, but from a different perspective since they propose a distinct policy.

5. MARGINAL UTILITY OF INCOME

In this section we obtain two properties of the individuals' marginal utilities of income. The first of them shows that, generically, the agents' ratios of marginal utilities across states do not coincide, a fact that is strictly derived from the incompleteness of markets. This fact also drives the result stated in the second property.

PROPERTY 1: *For a generic set of economies $\Gamma_5 \subset \Gamma$, at each CE, $\frac{\lambda^h(s)}{\lambda^h(s')} \neq \frac{\lambda^{h'}(s)}{\lambda^{h'}(s')}$ for every $h, h' \in \mathcal{H}$, $h \neq h'$, and for every $s, s' \in \mathcal{S}$, $s \neq s'$.*

PROOF: Define the set $M_R := \{m \in \Re^{S+1} : R^T \cdot m = \underline{0}\}$. Since, by Assumption (A): (iia) and (iic), $rank(R) = A+1$ and $S+1 > A+1$, we can conclude that M_R is generated by a vector space of dimension greater than or equal to one. For an arbitrary $\tilde{s} \in \mathcal{S}$, consider a subset of $A+1$ states $\hat{\mathcal{S}} \subset \mathcal{S} \setminus \{\tilde{s}\}$, ordered as $s_0, s_1, \cdots, s_A$, set $\hat{m}_s = 0$ for every $s \notin \hat{\mathcal{S}}$, and let $\hat{m}_{\tilde{s}} \neq 0$ be an arbitrary number. The equation $-\hat{m}_{\tilde{s}} \cdot r(\tilde{s}) = \sum_{s \in \mathcal{S}\setminus\{\tilde{s}\}} \hat{m}_s \cdot r(s)$ has a solution since, by Assumption (A): (iii), every set of $A+1$ vectors that can be extracted from the set $\{r(0), r(1), \cdots, r(S)\}$ is linearly independent so they span $\Re^{A+1}$. It follows that it is possible to pick a vector $\hat{m} \in M_R \setminus \{\underline{0}\}$ even though one coordinate is arbitrarily prespecified.

Fix an equilibrium allocation of a given economy $(\omega, u) \in \Gamma$, and consider, for an individual $h \in \mathcal{H}$, the Lagrange multipliers $\lambda^h(s)$, $s \in \mathcal{S}$, at the chosen equilibrium. Define the $S+1$ dimensional vector $\lambda^h := \Big(\lambda^h(s)\Big)_{s \in \mathcal{S}}$. Notice that $\mu^h \cdot \hat{q} = R^T \cdot \lambda^h$ specifies the condition obtained earlier for the optimal choice of an asset portfolio by individual h.

Consider two given individuals, $h, h' \in \mathcal{H}$, $h \neq h'$, and two given states $s, s' \in \mathcal{S}$, $s \neq s'$.

Perturb individual h's utility by adding a quantity denoted by $\tau_l(s)$ to each derivative $\frac{\partial u^h(x^h)}{\partial x_l^h(s)}$ at the equilibrium so as to perturb the vector λ^h by the addition of a vector $d\lambda^h$ and leave the conditions for agent h's optimal choice of goods unaffected. So, for a given state $s \in \mathcal{S}$, and for every $l \in \mathcal{L}$, the quantities $\tau_l(s)$ and $d\lambda^h(s)$ satisfy $\tau_l(s) = \hat{p}_l(s) \cdot d\lambda^h(s)$, given the equilibrium price $\hat{p}_l(s)$, and $d\lambda^h := \Big(d\lambda^h(s)\Big)_{s \in \mathcal{S}}$. The *new* vector of multipliers induced by the perturbation is $\lambda^h + d\lambda^h$.

By the properties of the set M_R, it is possible to choose a $d\lambda^h \in M_R$ such that $\begin{pmatrix} d\lambda^h(s) \\ d\lambda^h(s') \end{pmatrix} \neq \underline{0}$. This allows us to construct the utility perturbation described above. That perturbation does not affect the optimal choice of assets of individual h, given that

$$R^T \cdot (\lambda^h + d\lambda^h) = R^T \cdot \lambda^h + R^T \cdot d\lambda^h = R^T \cdot \lambda^h + \underline{0} = R^T \cdot \lambda^h.$$

Now define the matrix, of dimension 2×2,

$$Q := \begin{bmatrix} \lambda^h(s) & \lambda^{h'}(s) \\ \lambda^h(s') & \lambda^{h'}(s') \end{bmatrix},$$

and the function:
$\varphi : \Omega \times \mathcal{U} \times P \times Q \times \Delta^1 \to \Re^{L(S+1)} \times \Re^A \times \Re^2$, where Δ^1 is the one dimension simplex, and φ is specified as
$\varphi(\omega, u, \hat{p}, \hat{q}, \delta) := [(\hat{z}, \hat{\Theta})(\omega, u, \hat{p}, \hat{q}), \delta^T \cdot Q]$.

The perturbation of utilities specified above can be used to show that φ is transversal to zero. Since the dimension of the range of $\varphi_{\omega,u}$ exceeds that of the domain, by applying the Regular Value Theorem, we know that $\varphi^{-1}_{\omega,u}(\underline{0}) = \emptyset$ for every $(\omega, u) \in \Gamma_5$, where $\Gamma_5 \subset \Gamma$ is a generic set. Hence, for such a set of economies, there does not exist any $\delta \in \Delta^1$ such that $\delta^T \cdot Q = \underline{0}^T$, that is, the rank of the matrix Q is 2, and the result is obtained.

■

PROPERTY 2: *Given Property 1, given* $\delta := (\delta_{h_0}, \delta_{h_1}, \cdots, \delta_{h_L}) \in \Delta^L$ *such that* $\delta_{h_0} \neq 0$, *there exists a set of* $L+1$ *individuals,* $\{h_0, h_1, ..., h_L\} \subset \mathcal{H}$, *such that*
$0 \neq \delta_{h_0} \cdot \lambda^{h_0}(s) \neq \delta_{h_i} \cdot \lambda^{h_i}(s)$ *for at least* S *states, for every* $i \in \{1, 2..., L\}$, *for all* $(\omega, u) \in \Gamma_5$.

PROOF: Notice that, since we obtain only interior solutions, $\lambda^h(s) \neq 0$ for all $h \in \mathcal{H}$, and for all $s \in \mathcal{S}$ at a CE.

Consider an individual $h_0 \in \mathcal{H}$, a subset of states $\tilde{\mathcal{S}} \subset \mathcal{S}$ such that $\#\tilde{\mathcal{S}} := S$, and pick a $\delta := (\delta_{h_0}, \delta_{h_1}, \cdots, \delta_{h_L}) \in \Delta^L$ such that $\delta_{h_0} \neq 0$. By assuming that $H \geq 2L$, we are able to either:

(i) extract from $\mathcal{H} \setminus \{h_0\}$ a set of individuals $\{h_1, h_2, \cdots, h_L\} \subset \mathcal{H} \setminus \{h_0\}$ for which
$\delta_{h_i} \cdot \lambda^{h_i}(s) \neq \delta_{h_0} \cdot \lambda^{h_0}(s)$ for every $i \in \{1, 2..., L\}$, for every $s \in \tilde{\mathcal{S}}$, and, hence, Property 2 holds, or

(ii) extract from $\mathcal{H} \setminus \{h_0\}$ a set of individuals $\{h'_1, h'_2, \cdots, h'_L\} \subset \mathcal{H} \setminus \{h_0\}$ such that
$\delta_{h'_i} \cdot \lambda^{h'_i}(\bar{s}) = \delta_{h_0} \cdot \lambda^{h_0}(\bar{s})$, for every $i \in \{1, 2..., L\}$, for some $\bar{s} \in \tilde{\mathcal{S}}$. Then, by using the result stated in Property 1, we know that $\frac{\lambda^{h'_i}(\bar{s})}{\lambda^{h'_i}(s)} \neq \frac{\lambda^{h_0}(\bar{s})}{\lambda^{h_0}(s)}$ for every $i \in \{1, 2..., L\}$, for every $s \in \mathcal{S} \setminus \{\bar{s}\}$, for every $(\omega, u) \in \Gamma_5$. Therefore, by specifying the set $\breve{\mathcal{S}} := \mathcal{S} \setminus \{\bar{s}\}$, we obtain that $\delta_{h_i} \cdot \lambda^{h_i}(s) \neq \delta_{h_0} \cdot \lambda^{h_0}(s)$ for every $i \in \{1, 2..., L\}$, for every $s \in \breve{\mathcal{S}}$, for all $(\omega, u) \in \Gamma_5$, as required.

■

6. A RESULT FROM LINEAR ALGEBRA

This section presents a technical result from linear algebra that is used in the proof.

Consider a set of linearly independent vectors of dimension L, $\mathcal{V} = \{v_1, ..., v_L\}$, an arbitrary set of L numbers, $\{\alpha_1, \alpha_2, \cdots, \alpha_L\}$, and specify a set of vectors
$E := \{e_1, e_2, \cdots, e_L\}$, of dimension L, by setting $e_i := \alpha_i \cdot v_i$ for every $i \in \{1, 2, ..., L\}$.

Define the vector $\alpha := \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_L \end{pmatrix}$, of dimension L, and the matrix, of dimension $L \times L$, $\phi := [v_1 \;\; v_2 \;\; \cdots \;\; v_L]$.

Let the vector e be specified as $e := \sum_{i=1}^{L} e_i$. Since $\mathcal{V}$ spans $\Re^L$ and

$\sum_{i=1}^{L} e_i = \sum_{i=1}^{L} \alpha_i \cdot v_i = \phi \cdot \alpha$,

we know that any vector $e \in \Re^L$ can be generated by suitably choosing the vector α.

Now consider a set of non zero numbers $\{a_0, a_1, ..., a_L\}$ such that $a_0 \neq a_i$ for all $i \in \{1, 2, ..., L\}$, and define the matrix, of dimension $L \times L$,

$$A := \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_L \end{bmatrix}.$$

Given that A is diagonal, the set of its eigenvalues coincides with that of the matrix $\phi \cdot A \cdot \phi^{-1}$. Since $a_0 \neq a_i$ for all $i \in \{1, 2, ..., L\}$, it follows that a_0 is not an eigenvalue of A, and that it is not an eigenvalue of $\phi \cdot A \cdot \phi^{-1}$ either. Hence, $| \, a_0 \cdot I - \phi \cdot A \cdot \phi^{-1} \, | \neq 0$, where I is the identity matrix of dimension $L \times L$. So, the matrix $(a_0 \cdot I - \phi \cdot A \cdot \phi^{-1})$ has full rank.

Notice that the vector $(a_0 \cdot I - \phi \cdot A \cdot \phi^{-1}) \cdot e$ can be rewritten as:

$(a_0 \cdot I - \phi \cdot A \cdot \phi^{-1}) \cdot (\phi \cdot \alpha) = a_0 \cdot e - \phi \cdot A \cdot \alpha = a_0 \cdot \sum_{i=1}^{L} \alpha_i \cdot v_i - \sum_{i=1}^{L} a_i \cdot \alpha_i \cdot v_i$.

We have shown that

LEMMA (L): *Given a set of L non zero numbers $\{a_0, a_1, ..., a_L\}$ such that $a_0 \neq a_i$ for all $i \in \{1, 2, ..., L\}$, and a set of L linearly independent vectors of dimension L, $\{v_1, ..., v_L\}$, any vector $a_0 \cdot \sum_{i=1}^{L} \alpha_i \cdot v_i - \sum_{i=1}^{L} a_i \cdot \alpha_i \cdot v_i$, of dimension L, can be generated by suitably choosing the set of numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_L\}$.*

7. PROOF OF THE RESULT

In this section we provide the proof of Theorem (T) by making use of the various arguments presented up to now.

Let us specify the generic set of economies that are strongly regular, Proposition 2, and for which, at any CE, the *spanning property*, the property stated in Subsection 4.2, and Property 2 are satisfied, as $\hat{\Gamma} := \cap_{k=2}^{5} \Gamma_k$.

Consider a CE, with equilibrium prices $(\hat{p}, \hat{q})$, of a given economy $(\omega, u) \in \hat{\Gamma}$. Let us recall that the key procedure to prove Theorem (T) is to show that the matrix C defined in Section 3 has full rank for a generic set of economies. Since we are interested in proving a generic feature, we need to perturb the economy (ω, u). We do this by setting an additive perturbation that induces (ω, u) to move to a neighbouring economy, that is,

$$(\omega, u) \longmapsto (\omega, u) + (\Delta\omega, \Delta u),$$

where $\Delta\omega$ and Δu denote, respectively, the perturbation to endowments and the perturbation to utilities.

Let us describe first the perturbation to endowments.

Consider a set of $L+1$ individuals $\{h_0, h_1, ..., h_L\} \subset \mathcal{H}$ and a subset of states $\tilde{\mathcal{S}} \subset \mathcal{S}$, $\#\tilde{\mathcal{S}} = S$, ordered as $s_1, \cdots, s_S$. Set $\{\bar{s}\} := \mathcal{S}/\tilde{\mathcal{S}}$. For the moment both the sets, that of individuals and of states, are arbitrary. Consider, for every $s \in \tilde{\mathcal{S}}$, an arbitrary set of numbers $\{\gamma_1(s), \gamma_2(s), \cdots, \gamma_L(s)\}$. Let the vector $\Delta\omega$ be specified as:

(a) $\Delta\omega^h := \underline{0}$ for every $h \notin \{h_0, h_1, ..., h_L\}$,

(b) for every $i \in \{1, 2, ..., L\}$,

$$\Delta\omega^{h_i}(s) = \begin{pmatrix} \Delta\omega_0^{h_i}(s) \\ \Delta\omega_1^{h_i}(s) \\ \vdots \\ \Delta\omega_L^{h_i}(s) \end{pmatrix} := \begin{pmatrix} \Delta\omega_0^{h_i}(s) \\ \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right] \end{pmatrix},$$

for every $s \in \tilde{\mathcal{S}}$, and $\Delta\omega^{h_0}(\bar{s}) := \underline{0}$,

(c) $$\Delta\omega^{h_0}(s) = \begin{pmatrix} \Delta\omega_0^{h_0}(s) \\ \Delta\omega_1^{h_0}(s) \\ \vdots \\ \Delta\omega_L^{h_0}(s) \end{pmatrix} := \begin{pmatrix} \Delta\omega_0^{h_0}(s) \\ -\sum_{i=1}^{L} \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right] \end{pmatrix}$$

for every $s \in \tilde{\mathcal{S}}$, and $\Delta\omega^{h_0}(\bar{s}) := \underline{0}$.

In addition, $\Delta\omega_0^{h_i}$, $i \in \{1, 2, ..., L\}$, and $\Delta\omega_0^{h_0}$ are specified as to satisfy the constraints:

for every $i \in \{0, 1, 2, ..., L\}$, $\quad \Delta\omega_0^{h_i}(s) + \sum_{l \in \mathcal{L} \setminus \{0\}} \hat{p}_l(s) \cdot \Delta\omega_l^{h_i}(s) = 0 \quad$ for every $s \in \tilde{\mathcal{S}}$,
so that income remains unaffected.

Let us denote by $\Delta\hat{z}^h$ the change induced in individual h's excess demand by the perturbation of endowments. Also, denote the change induced in aggregate excess demand by $\Delta\hat{z}$.

We know that the perturbation to endowments does not change the optimal choices of any individual since it leaves unaffected the individuals' budget contraints at every state. Also, it permits us to obtain that

$\Delta\hat{z}^h = \underline{0}$ for every $h \notin \{h_0, h_1, ..., h_L\}$,

$\Delta\hat{z}^{h_i}(s) = \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right] \quad$ for every $i \in \{1, 2, ..., L\}$ and for every $s \in \tilde{\mathcal{S}}$,

$\Delta\hat{z}^{h_0}(s) = -\sum_{i=1}^{L} \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right]$ for every $s \in \tilde{\mathcal{S}}$,

$\Delta\hat{z}^{h_i}(\bar{s}) = \underline{0}$ for every $i \in \{0, 1, 2, ..., L\}$.

For $i \in \{0, 1, 2, ..., L\}$, let $\Delta\hat{z}^{h_i} = \Big(\Delta\hat{z}^{h_i}(s)\Big)_{s\in\mathcal{S}}$, an $L(S+1)$ dimensional column vector.

For an arbitrary vector $\delta := (\delta_h)_{h\in\mathcal{H}} \in \Delta^H$ we analyze the change in $\delta^T \cdot (\widetilde{\lambda \odot \hat{z}})$, a row vector with $L(S+1)$ components, of which the L components corresponding to the state $\bar{s}$ are zero, induced by the specified perturbation to endowments.

$$\delta^T \cdot (\widetilde{\lambda \odot \Delta\hat{z}}) = \textstyle\sum_{h\in\mathcal{H}} \delta_h \cdot (\lambda^h \odot \Delta\hat{z}^h) = \delta_{h_0} \cdot (\lambda^{h_0} \odot \Delta\hat{z}^{h_0}) + \sum_{i=1}^L \delta_{h_i} \cdot (\lambda^{h_i} \odot \Delta\hat{z}^{h_i}),$$

since $\Delta\hat{z}^h = \underline{0}$ for every $h \notin \{h_0, h_1, ..., h_L\}$. It follows that δ_h, $h \notin \{h_0, h_1, ..., h_L\}$, play no role so, without loss of generality, we can consider $\delta \in \Delta^L$. Upon substituting for $\Delta\hat{z}^{h_i}$ we obtain

$$\begin{aligned}
\delta^T \cdot (\widetilde{\lambda \odot \Delta\hat{z}}) = & -\delta_{h_0} \cdot \Big(\lambda^{h_0}(s_1) \cdot \Big[\textstyle\sum_{i=1}^L \gamma_i(s_1) \cdot [V^{h_i}(s_1) - V^{h_0}(s_1)]\Big]^T \quad \cdots \quad \Big[\underline{0}\Big]^T \quad \cdots \\
& \cdots \quad \lambda^{h_0}(s_S) \cdot \Big[\textstyle\sum_{i=1}^L \gamma_i(s_S) \cdot [V^{h_i}(s_S) - V^{h_0}(s_S)]\Big]^T\Big) \\
& + \textstyle\sum_{i=1}^L \delta_{h_i} \cdot \Big(\lambda^{h_i}(s_1) \cdot \gamma_i(s_1) \cdot \Big[V^{h_i}(s_1) - V^{h_0}(s_1)\Big]^T \quad \cdots \quad \Big[\underline{0}\Big]^T \quad \cdots \\
& \cdots \quad \lambda^{h_0}(s_S) \cdot \gamma_i(s_S) \cdot \Big[V^{h_i}(s_S) - V^{h_0}(s_S)\Big]^T\Big) \\
= & \Big(-\delta_{h_0} \cdot \lambda^{h_0}(s_1) \cdot \Big[\textstyle\sum_{i=1}^L \gamma_i(s_1) \cdot [V^{h_i}(s_1) - V^{h_0}(s_1)]\Big]^T \\
& + \textstyle\sum_{i=1}^L \delta_{h_i} \cdot \lambda^{h_i}(s_1) \cdot \gamma_i(s_1) \cdot \Big[V^{h_i}(s_1) - V^{h_0}(s_1)\Big]^T \\
& \cdots \quad \Big[\underline{0}\Big]^T \quad \cdots \\
& -\delta_{h_0} \cdot \lambda^{h_0}(s_S) \cdot \Big[\textstyle\sum_{i=1}^L \gamma_i(s_S) \cdot [V^{h_i}(s_S) - V^{h_0}(s_S)]\Big]^T \\
& + \textstyle\sum_{i=1}^L \delta_{h_i} \cdot \lambda^{h_i}(s_S) \cdot \gamma_i(s_S) \cdot \Big[V^{h_i}(s_S) - V^{h_0}(s_S)\Big]^T\Big)
\end{aligned}$$

so that there are $S+1$ blocks of L dimensional row vectors of which one block, the one corresponding to state $\bar{s}$, is a vector of zeroes.

Let δ be such that $\delta_{h'} > 0$ for some $h' \in \mathcal{H}$. Use Property 2 of Section 5 to specify a set of $L+1$ individuals, denoted $\{h_0, h_1, ..., h_L\}$, and a set of states $\tilde{\mathcal{S}}$, such that $0 \neq \delta_{h_0} \cdot \lambda^{h_0}(s) \neq \delta_{h_i} \cdot \lambda^{h_i}(s)$ for all $s \in \tilde{\mathcal{S}}$. Use the specified set of individuals and the set $\tilde{\mathcal{S}}$ of states to construct the endowment perturbation specified above with $\{\gamma_1(s), \gamma_2(s), \cdots, \gamma_L(s)\}$, $s \in \tilde{\mathcal{S}}$, being arbitrary numbers. For each $s \in \tilde{\mathcal{S}}$ apply Lemma (L) with $\delta_{h_i} \cdot \lambda^{h_i}(s)$ playing the role of a_i, $i = 0, 1, \cdots, L$, $\{\gamma_1(s), \gamma_2(s), \cdots, \gamma_L(s)\}$ playing the role of $\{\alpha_1, \alpha_2, \cdots, \alpha_L\}$, and $\{V^{h_1}(s) - V^{h_0}(s), V^{h_2}(s) - V^{h_0}(s), \cdots, V^{h_L}(s) - V^{h_0}(s)\}$ playing the role of $\{v_1, ..., v_L\}$; the Lemma can be applied by invoking the spanning property obtained in Subsection 4.1. It follows that any vector $\delta^T \cdot (\widetilde{\lambda \odot \Delta\hat{z}})$ with LS non zero coordinates can be generated by suitably picking the set of numbers $\{\gamma_1(s), \gamma_2(s), \cdots, \gamma_L(s)\}$ for every $s \in \tilde{\mathcal{S}}$ since LS coordinates can be controlled independently.

The perturbation of endowments specified above also changes the matrix $D_{\hat{p}}\hat{z}$ which we now analyze. Consider a given state $s \in \tilde{\mathcal{S}}$. Let us denote by $\Delta D_{\hat{p}(s)}\hat{z}^h(s)$, $h \in \mathcal{H}$, and by $\Delta D_{\hat{p}(s)}\hat{z}(s)$, the changes induced, respectively, in matrices $D_{\hat{p}(s)}\hat{z}^h(s)$, $h \in \mathcal{H}$, and $D_{\hat{p}(s)}\hat{z}(s)$, by the perturbation of endowments. For an individual $h \in \mathcal{H}$, the Slutsky decomposition of the matrix $D_{\hat{p}(s)}\hat{z}^h(s)$ gives:[5]

$$D_{\hat{p}(s)}\hat{z}^h(s) = \lambda^h(s) \cdot K^h(s) - V^h(s) \cdot [\hat{z}^h(s)]^T,$$

where $K^h(s)$ is a symmetric matrix of dimension $L \times L$.

We remark that, for any $h \in \mathcal{H}$ and for any $s \in \mathcal{S}$, $\lambda^h(s)$, $K^h(s)$, and $V^h(s)$ are not affected by the specified perturbation of endowments since income, and hence demand, remains unaffected. Now, by making use of the induced changes to individuals excess demands, $\Delta\hat{z}^h(s)$, and the fact that $D_{\hat{p}(s)}\hat{z}(s) = \sum_{h\in\mathcal{H}} D_{\hat{p}(s)}\hat{z}^h(s)$ for every $s \in \mathcal{S}$, we obtain that

$$\Delta D_{\hat{p}(s)}\hat{z}(s) = -\sum_{i=0}^{L} V^{h_i}(s) \cdot \left[\Delta\hat{z}^{h_i}(s)\right]^T =$$
$$-V^{h_0}(s)\sum_{i=1}^{L} \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right]^T + \sum_{i=1}^{L} V^{h_i}(s) \cdot \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right]^T =$$
$$\sum_{i=1}^{L} \gamma_i(s) \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right] \cdot \left[V^{h_i}(s) - V^{h_0}(s)\right]^T.$$

To ease the notational burden, let us relabel each coordinate $\left[V_l^{h_i}(s) - V_l^{h_0}(s)\right]$ as $A_l^{h_i}(s)$ for every $i \in \{1, 2, ..., L\}$, and for every $l \in \mathcal{L} \setminus \{0\}$. By writing out the product above, we derive the matrix of dimension $L \times L$

$$\Delta D_{\hat{p}(s)}\hat{z}(s) =$$
$$\begin{bmatrix} \sum_{i=1}^{L} \gamma_i(s) \cdot A_1^{h_i}(s) \cdot A_1^{h_i}(s) & \sum_{i=1}^{L} \gamma_i(s) \cdot A_1^{h_i}(s) \cdot A_2^{h_i}(s) & \cdots & \sum_{i=1}^{L} \gamma_i(s) \cdot A_1^{h_i}(s) \cdot A_L^{h_i}(s) \\ \sum_{i=1}^{L} \gamma_i(s) \cdot A_2^{h_i}(s) \cdot A_1^{h_i}(s) & \sum_{i=1}^{L} \gamma_i(s) \cdot A_2^{h_i}(s) \cdot A_2^{h_i}(s) & \cdots & \sum_{i=1}^{L} \gamma_i(s) \cdot A_2^{h_i}(s) \cdot A_L^{h_i}(s) \\ \vdots & \vdots & & \vdots \\ \sum_{i=1}^{L} \gamma_i(s) \cdot A_L^{h_i}(s) \cdot A_1^{h_i}(s) & \sum_{i=1}^{L} \gamma_i(s) \cdot A_L^{h_i}(s) \cdot A_2^{h_i}(s) & \cdots & \sum_{i=1}^{L} \gamma_i(s) \cdot A_L^{h_i}(s) \cdot A_L^{h_i}(s) \end{bmatrix}, \quad (7)$$

which happens to be symmetric.

Let us now introduce the perturbation to utilities, Δu. Consider an individual $\tilde{h} \in \mathcal{H}$, and construct Δu by placing a quadratic term, that we now describe, in the coordinate that corresponds to individual $\tilde{h}$, and by placing zeroes in the other coordinates. This quadratic term is such that the linear term subsequently added to the vectors of first derivatives of $u^{\tilde{h}}$ amounts to zero at the CE; hence, it leaves aggregate demand unaffected,

[5] See, for instance, Geanakoplos and Polemarchakis (1980).

but changes the matrix of second derivatives of $u^{\tilde{h}}$.[6] In addition, this quadratic term induces, for every $s \in \mathcal{S}$, a change in the matrix $K^{\tilde{h}}(s)$ by the addition of a symmetric matrix that cancels out with the matrix above, equation (7).

We turn to analyze the effects of the perturbations $(\Delta\omega, \Delta u)$ on the vector $\delta^T \cdot C$ where δ is chosen arbitrarily and the perturbation depends on δ. Denote by ΔC the change induced in the matrix C. Since the specified perturbations do not change the matrix $D_{\hat{p}}\hat{z}$, we obtain

$$\delta^T \cdot \Delta C = \delta^T \cdot (\widetilde{\lambda \odot \Delta \hat{z}}) \cdot (D_{\hat{p}}\hat{z})^{-1} \cdot \Big[(V^1 - V^0) \odot r_0 \quad \cdots \quad (V^H - V^0) \odot r_0 \quad (V^1 - V^0) \odot r_1\Big].$$

Also, as noted earlier, we know that the vector $\delta^T \cdot (\widetilde{\lambda \odot \Delta \hat{z}})$ can be generated as desired for at least LS of the coordinates; hence, so can the vector $(\widetilde{\lambda \odot \Delta \hat{z}}) \cdot (D_{\hat{p}}\hat{z})^{-1}$ given that we are able to leave unaffected the matrix $D_{\hat{p}}\hat{z}$. The property stated in Subsection 4.2 assures us that the matrix obtained by eliminating those rows of the matrix $[(V^1 - V^0) \odot r_0 \quad \cdots \quad (V^H - V^0) \odot r_0 \quad (V^1 - V^0) \odot r_1]$ that correspond to some dropped state, has rank $H+1$, and hence has at least $H+1$ linearly independent rows. It follows that, given δ, by specifying a perturbation that generates non zero entries only in those components of $\delta^T \cdot (\widetilde{\lambda \odot \Delta \hat{z}}) \cdot (D_{\hat{p}}\hat{z})^{-1}$ that correspond to some set of $H+1$ linearly independent rows, we can guarantee that $\delta^T \cdot \Delta C \neq \underline{0}^T$.

Consequently, by applying a transversality argument, we obtain that $\delta^T \cdot C \neq \underline{0}^T$ for every $(\omega, u) \in \tilde{\Gamma}$, where $\tilde{\Gamma} \subset \hat{\Gamma}$ is a generic set.

Since δ was chosen arbitrarily, it follows that the matrix C has rank $H+1$ for a generic set of economies $\tilde{\Gamma}$. This completes the proof of the Theorem (T).

REMARK 1: The GP result holds for a generic set of economies. Of course, there are non-generic economies for which some CE are not CS. As in GP, consider an economy $(\omega, u) \in \Gamma$ for which there is a CE such that no agent trades any good at any state, that is, $\hat{z}^h_l(s) = 0$ for all $h \in \mathcal{H}$, all $l \in \mathcal{L}$, and all $s \in \mathcal{S}$. If this is the case, then, for every $h \in \mathcal{H}$, the last sum in equation (3) amounts to zero, and, hence, the contribution to the change of utility of any agent due to the change in relative prices vanishes. It follows that, given a reallocation of asset holdings $d\theta$, $D_\theta u$ only captures the effect of a pure redistribution of income so that no improvement can be induced. However, we know that the economy (ω, u) belongs to a non-generic set since, by changing slightly the parameter ω, we move to a new economy such that some individuals trade at each CE, which implies that the set that contains (ω, u) is not open.

REMARK 2: GP also show that requiring that the asset reallocation $\widetilde{d\theta}$ be budget feasible for every agent, $\sum_{a \in \mathcal{A}} \hat{q}_a \cdot d\check{\theta}^h = 0$ for all $h \in \mathcal{H}$, eliminates the direct effect on individuals' utilities. We recall that $\hat{q}_a = \frac{1}{\mu^h} \cdot \sum_{s \in \mathcal{S}} r_a(s) \cdot \lambda^h(s)$ is the condition for the optimal choice

[6] It is well known that by adding a suitable quadratic term to u^h, one can induce any perturbation of the matrix $K^h(s)$, $h \in \mathcal{H}$, $s \in \mathcal{S}$, by the addition of a symmetric matrix. See, for instance, Geanakoplos and Polemarchakis (1980).

of asset a by individual h, which enables us to rewrite the said feasibility requirement as $\frac{1}{\mu^h} \cdot \sum_{a \in \mathcal{A}} \sum_{s \in \mathcal{S}} r_a(s) \cdot \lambda^h(s) \cdot d\check{\theta}^h = 0$ for all $h \in \mathcal{H}$; it follows that the first vector in equation (6) is equal to $\underline{0}$. But imposing budget feasibility has no consequence for the relative price effect, and, hence, the GP result continues to apply through it.

REMARK 3: One wants to know whether the bound on the number of agents is tight. As we now argue, if $LS < H + 1 \leq L(S+1)$ then the argument given to prove the result fails to hold. We recall that Property 2, combined with the result from linear algebra given in Section 6, only guarantees that, given a $\delta \in \Delta^H$, the vector $\delta^T \cdot (\widetilde{\lambda \odot \Delta\hat{z}})$ can be generated as desired for LS components. It follows that to show that matrix C has full rank, the set of vectors $\{(V^1 - V^0) \odot r_0, \cdots, (V^H - V^0) \odot r_0, (V^1 - V^0) \odot r_1\}$ need to be linearly independent when considering any LS coordinates of them, which can be achieved only if $H + 1 \leq LS$.

REMARK 4: Geanakoplos, Magill, Quinzii, and Dreze (1990) extend the result of constrained suboptimality to the case of an economy with production. The key argument in their proof resembles the one given here in that they show that, generically, there is no vector δ such that $\delta^T \cdot Q = \underline{0}^T$, where Q is a matrix whose entries reflect the changes of the prices of products due to changes in the level of production. So, they too focus on analysing the influence of the price effect of a redistribution of assets and goods on welfare. However, that is done from the perspective of the supply side of the economy, so that instead of perturbing utilities they perturb endowments and production plans.

8. CONCLUSIONS

The role of the incompleteness of markets in generating the constrained subotimality of CE of pure exchange economies depends crucially on the relative price effect on utilities induced by an asset reallocation. The difficulty in proving the result for a generic set of economies is in constructing a perturbation of the individuals' endowments that is related to the proposed asset reallocation in a way such that the price effect can be clearly analyzed.

If the intervention considered does not allow for a direct control of the individuals' income effects, then an upper bound needs to be imposed on the number of agents to achieve the result.

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