The Unit Root Property when Markets Are Sequentially Incomplete*

Subir CHATTOPADHYAY
Universidad de Alicante, and
El Colegio de México
e-mail: subir@merlin.fae.ua.es

and

Antonio JIMENEZ MARTINEZ
Universidad de Alicante
e-mail: jimenez@merlin.fae.ua.es

First Version: March 2000
Revised: January 2004

Correspondence To:
S. Chattopadhyay
Departamento de Fundamentos del Análisis Económico
Universidad de Alicante
E-03071 Alicante, SPAIN.
Telephone: 00-34-96-590-3563
Fax: 00-34-96-590-3685
e-mail: subir@merlin.fae.ua.es

* We thank H. M. Polemarchakis for useful discussions. Chattopadhyay gratefully acknowledges financial support from CONACYT via a Cátedra de Excelencia at El Colegio de México, additional support from the IVIE and DGICyT project PB97-0131, and the hospitality of ColMex; Jimenez Martinez acknowledges support from the DGICyT in the form of a Doctoral Fellowship, additional support from the IVIE and the hospitality of CORE and ColMex where part of the research was carried out.

ABSTRACT

We consider pure exchange, one good OLG economies under stationary Markov uncertainty. It is known that when markets are sequentially complete, a stationary equilibrium at which the agents common matrix of intertemporal rates of substitution has a Perron root which is less than or equal to one is conditionally Pareto optimal (CPO). We assume that there exists a long-lived dividend paying asset and show that if dividends are strictly positive then the relation between the unit root condition and a constrained notion of optimality holds even if markets are not sequentially complete. However, every equilibrium allocation is shown to be constrained CPO under the additional requirement that assets be freely disposable, which seems reasonable when dividends are positive and whose importance was pointed out by Santos and Woodford (1997) in their work on bubbles; this fact undermines the relation between the unit root property and optimality. The relation is less clear when dividends and asset prices are allowed to be negative in some states.

Journal of Economic Literature Classification Numbers: D52, D61

1. INTRODUCTION

In overlapping generations (OLG) economies with sequential trading, the demographic structure of the model severely restricts the possibilities for trade. These restrictions become even more relevant when one considers economies with uncertainty. The study of intertemporal risk sharing in these OLG models is of interest because it provides a framework for the debate on some macroeconomic issues, social security systems being the most important example.

It is known that when enough markets exist to let agents insure against all risks that arise after their birth, so that markets are *sequentially complete*, a stationary equilibrium is optimal if and only if it satisfies the *unit root property*, i.e., the Perron root of every agent's matrix of marginal rates of substitution is less than or equal to one. The notion of optimality used is that of conditional Pareto optimality (CPO) proposed by Muench (1977) in which the Pareto criterion is applied in an environment in which agents are distinguished by their state of birth in addition to the date of their birth and their type.

Our objective is to analyze the optimal allocation of risk in OLG economies where markets are potentially not even sequentially complete, i.e., where agents are unable to insure themselves completely against even those sources of uncertainty which arise after their birth. It is easy to see that if there are no long-lived assets in the economy then it is impossible to have non-zero net intergenerational transfers and as a consequence the OLG structure becomes redundant. So we consider economies in which there exists a "tree", i.e., an asset which pays a dividend in each state and which can be retraded repeatedly (the dividend can be negative and could also be identically zero). We provide separate necessary and sufficient conditions on the configuration of dividends and asset prices under which equilibria are optimal. We are able to relate these conditions to the unit root property when dividends are positive.

We consider a simple environment with stationary Markov uncertainty and only one consumption good. Since we wish to analyze the optimality properties of equilibria, we need to specify a notion of optimality which is appropriate for our purposes; we propose a definition which takes into account the fact that markets may be sequentially incomplete by requiring that consumption possibilities when old be determined by the returns generated by the set of assets that are available. To be precise, optimality of an allocation is gauged against alternatives in which consumption when young and an asset portfolio are allocated directly, while consumption when old is specified via the return on the portfolio and the agent's endowment. Since there is only one good, this scheme completely specifies consumption possibilities once the return on assets is determined. We will assume that assets earn their equilibrium return even though the portfolio allocation has been changed.[1] An allocation can be constrained improved if there exists a constrained feasible allocation in the sense just described which is also a CPO improvement.

[1] So we do not consider the price effects that form an essential part of the analysis of generic constrained suboptimality of two period multi-good general equilibrium models with incomplete markets as in Geanakoplos and Polemarchakis (1986).

We show that if the stationary equilibrium is one in which the dividend is strictly positive in every state in which the price of the asset plus the dividend is negative, then the allocation can be improved upon by reallocating the existing assets in a stationary way. On the other hand we show that a stationary equilibrium allocation is constrained CPO if the sum of the price of the long-lived asset and its dividend is non-negative in every state and is strictly positive in some state; this condition implies that the price of the asset is always positive. In particular, imposing non-negativity of the dividend process and free disposal of the long-lived asset guarantees that the equilibrium allocation is constrained CPO.

Regarding the relation between the unit root property and the results obtained, we are able to show that when dividends are non-negative our sufficient condition for constrained CPO is also sufficient to imply that the Perron root is less than one. Similarly, with non-negative dividends (but not zero in every state), our sufficient condition for constrained suboptimality implies that the equilibrium prices of the long-lived asset have negative sign in some states which, as we show, implies that the Perron root exceeds one. The role of the unit root property in determining the efficiency properties of equilibrium seems to be severely limited when more general dividend processes are allowed for.

Turning to the literature, recall that Wilson (1981) showed that in an infinite horizon economy the presence of an individual who owns a non-negligible fraction of the total endowment forces the value of the aggregate endowment to be finite; it follows that competitive equilibria are Pareto optimal. Scheinkman (1980) argued that the presence of an asset which is freely disposable and which pays a dividend in every period would ensure Pareto optimality in OLG economies under certainty; such as asset is a way of implementing Wilson's condition in a sequence economy. Santos and Woodford (1997) provide a direct generalization of Wilson's result and show that in a multi-good model with sequentially complete markets, all equilibria are CPO provided that the dividend from the long-lived asset is a *significant* proportion of the endowment at each date-event.[2]

A more special model appears in Allen and Gale (1997) who consider an economy with one good and one agent in every period, so that markets are effectively sequentially complete, and show that the introduction of a dividend paying asset into this special model implies that the resulting stationary equilibrium allocation is CPO.

Two other papers have asked related questions. Following the lines of Geanakoplos and Polemarchakis (1986), Cass, Green, and Spear (1992) considered the stationary monetary

[2]More generally, by invoking the "no arbitrage" property of asset prices one easily shows that the sum of the agents' budget constraints weighted by the Arrow price at the date-event at which they are born (we use the fact that with one good the optimization problem faced by each agent can be written as one with a single budget constraint even though markets are incomplete) is the same as the sum of the planner's feasibility constraints weighted by the Arrow price at the node. Lemma 2.4 in Santos and Woodford (1997) can be used to show that a strictly positive dividend and free disposal of the asset implies that a stationary equilibrium allocation has finite value under every possible Arrow pricing process. It follows that every stationary equilibrium allocation is constrained CPO. However, our sufficient condition for constrained CPO appears to be substantially weaker since it allows for zero dividends.

equilibria of a one-good stochastic OLG economy with incomplete asset markets and showed that there are no locally improving stationary redistributions of the one period lived assets when the price of money and the agents' money demands are allowed to adjust in response to the redistribution. The equilibrium that they consider displays the unit root property. Gottardi (1996) considers a model which is similar to the one in Cass, Green, and Spear (1992), and stationary redistributions of the sort we consider; he ignores the welfare of the initial old which is the crucial difference between his approach and ours.

A complete characterization (in terms of dividends and asset prices) of the constrained optimality properties of equilibria in OLG economies with sequential trading is available in Chattopadhyay (2003). The case with price effects, as in Geanakoplos and Polemarchakis (1986), is of considerable interest; however, no general result is available so far.[3] The case with more than one good is yet to be analysed.

The rest of the paper is structured as follows. Section 2 presents the model and notation. In Section 3 we briefly discuss the unit root property and relate it to the description of the economy. Section 4 presents our main results on constrained optimality of stationary equilibria of the one-good economy when markets fail to be sequentially complete and Section 5 concludes with a discussion of the results.

2. THE MODEL

We consider a one good, two period lifetime, pure exchange overlapping generations (OLG) economy under stationary Markov uncertainty. We turn to a formal description of the model and the notation used.

Time is discrete and dates are denoted by $t = 1, 2, 3, \cdots$.

Let $\mathcal{S}$ be the *state space* of the Markov process with $S := \#\mathcal{S} < \infty$. The structure of the *date-event tree* induced by all possible realizations of states from an initial date $t = 0$ is as follows. The *root* of the tree is $\sigma_0 \in \mathcal{S}$; the set of *nodes at date* t is denoted Σ_t where we set $\Sigma_1 := \{\sigma_0\} \times \mathcal{S}$, and, iteratively, set $\Sigma_t := \Sigma_{t-1} \times \mathcal{S}$ for $t = 2, 3, \cdots$. Define $\Sigma := \cup_{t \geq 1} \Sigma_t$ and $\Gamma := \{\sigma_0\} \cup \Sigma$. Elements of Γ are called *nodes* (to be thought of as the "date-events" or simply "events"), and a generic node is denoted by σ. Given a node $\sigma \in \Sigma$, $t(\sigma)$ denotes the value of t at which $\sigma \in \Sigma_t$, and $s(\sigma)$ identifies the Markov state. Clearly, a node $\sigma \in \Sigma_t$ is nothing but a string $(\sigma_0, s_1, s_2, \cdots, s_t)$, where $s_\tau \in \mathcal{S}$ denotes the realization of the process at date τ, $\tau = 1, \cdots, t$ (σ_0 is the realization at the initial date). It follows that the *predecessor* of a node $\sigma \in \Sigma_t$ is uniquely defined; it will be denoted by σ_{-1}, an element of Σ_{t-1}. The set of immediate successor nodes of a node σ is denoted σ^+.

One commodity is available for consumption at each node $\sigma \in \Sigma$.

At each node $\sigma \in \Sigma$, $\mathcal{H}$, a generation of agents, is born, where $H := \#\mathcal{H}$. Each agent lives at two dates. The consumption plan of an agent specifies the level of consumption in the event at birth and in its immediate successor nodes. A member of generation σ of type $h \in \mathcal{H}$ is denoted by (σ, h).

[3] See Demange (2002) for the special case with a single dividend paying asset with a uniformly positive dividend.

In addition, there is a set of H one period lived agents who enter the economy at each node $\sigma \in \Sigma_1$ at date 1; they constitute the generation of the "initial old", and are indexed by (σ, h, o), where $\sigma \in \Sigma_1$.

We will assume that the economy is stationary, i.e., that the characteristics (consumption sets, endowments, and utility functions) of each agent depend only on the realizations of the Markov state during her lifetime, not on time nor on past realizations. So, for any $(\sigma, \widehat{\sigma}) \in \Sigma \times \Sigma$, $s(\sigma) = s(\widehat{\sigma})$ implies that (i) for consumption sets $X_{\sigma,h} = X_{\hat{\sigma},h} := X_{s(\sigma),h}$, (ii) for endowments $\omega(\sigma, h) = \omega(\widehat{\sigma}, h) := \omega(s(\sigma), h)$, where $\omega(s, h) = (\omega(s; s, h), (\omega(s, s'; s, h))_{s' \in \mathcal{S}})$ describes the endowment at birth and in all successor nodes, and (iii) for utility functions $u_{\sigma,h} = u_{\hat{\sigma},h} := u_{s(\sigma),h}$ (for the initial old we use the notation $X_{s(\sigma),h,o}$, $\omega(s(\sigma); h, o)$, and $u_{s(\sigma),h,o}$). Let $\omega(s; h, o) = \omega(\tilde{s}, s; \tilde{s}, h)$ for all $s \in \mathcal{S}$ and for all $h \in \mathcal{H}$ for some $\tilde{s} \in \mathcal{S}$; this lets us introduce the initial old in a manner which is compatible with the stationary structure of the rest of the economy.

A consumption plan for agent (σ, h) will be denoted by $x(\sigma, h) = (x(\sigma; \sigma, h), (x(\sigma'; \sigma, h))_{\sigma' \in \sigma^+})$ ($x(\sigma; h, o)$ for the initial old); this notation allows us to consider nonstationary consumption plans even though the environment is stationary.

There is a set $\mathcal{J}$ of *one period lived* assets, where $\#\mathcal{J} =: J \leq S - 1$, with stationary payoffs (per unit) in the commodity described by $((r^j_s)_{s \in \mathcal{S}}) \in R^S$. Let $r^j := (r^j_1, \cdots, r^j_S)$. Since they are one period lived, it is natural to suppose that their total endowment is zero, i.e., they are *inside assets*.

There is also a *dividend paying asset* with stationary payoff (per unit) specified by the vector $((d_s)_{s \in \mathcal{S}}) \in R^S$. Let $d := (d_1, \cdots, d_S)$.

Only the initial old are endowed with the asset and their endowment of the asset is denoted by $\omega^{\mathrm{d}}(s(\sigma); h, o)$. We will assume that $\sum_{h \in \mathcal{H}} \omega^{\mathrm{d}}(s(\sigma); h, o) = 1$ for all $s \in \mathcal{S}$.

Denoting by $\omega(\sigma)$ the total endowment at node σ, we have:

$\omega(\sigma) := \sum_{h \in \mathcal{H}} \omega(s(\sigma); s(\sigma), h) + \sum_{h \in \mathcal{H}} \omega(s(\sigma); h, o) + 1 \cdot d_{s(\sigma)}$ for $\sigma \in \Sigma_1$,

$\omega(\sigma) := \sum_{h \in \mathcal{H}} \omega(s(\sigma); s(\sigma), h) + \sum_{h \in \mathcal{H}} \omega(s(\sigma_{-1}), s(\sigma); s(\sigma_{-1}), h) + 1 \cdot d_{s(\sigma)}$ for $\sigma \in \cup_{t \geq 2} \Sigma_t$.

We impose the following standard conditions:

ASSUMPTION 1:

(i) $1 \leq H < \infty$ and $1 \leq S < \infty$.

(iia) For all $(s, h, o) \in \mathcal{S} \times \mathcal{H}$, $X_{s,h,o} = R_+$, $u_{s,h,o} : X_{s,h,o} \to R$ is strictly monotone.

(iib) For all $(s, h) \in \mathcal{S} \times \mathcal{H}$, $X_{s,h} = R^{1+S}_+$, $\omega(s; s, h) \in R_{++}$ and $((\omega(s, s'; s, h))_{s' \in \mathcal{S}}) \in R^S_+/\{\underline{0}\}$, $u_{s,h} : X_{s,h} \to R$ is C^2, strictly monotone, and differentiably strictly quasi-concave.

(iiia) $\omega(s; h, o) = \omega(\tilde{s}, s; \tilde{s}, h)$ for all $s \in \mathcal{S}$ and for all $h \in \mathcal{H}$ for some $\tilde{s} \in \mathcal{S}$.

(iiib) $\omega^{\mathrm{d}}(s(\sigma); h, o) \in R_+$ for all $s \in \mathcal{S}$ and for all $h \in \mathcal{H}$ and $\sum_{h \in \mathcal{H}} \omega^{\mathrm{d}}(s(\sigma); h, o) = 1$ for all $s \in \mathcal{S}$.

(iv) For all $\sigma \in \Sigma$, $\omega(\sigma) \in R_{++}$.

DEFINITION 1: A *feasible allocation* x is given by an array $((x(\sigma;h,o))_{(\sigma,h)\in\Sigma_1\times\mathcal{H}},\ (x(\sigma,h))_{(\sigma,h)\in\Sigma\times\mathcal{H}})$ such that $x(\sigma;h,o)\in X_{s(\sigma),h,o}$ for all $(\sigma,h)\in\Sigma_1\times\mathcal{H}$, $x(\sigma,h)\in X_{s(\sigma),h}$ for all $(\sigma,h)\in\Sigma\times\mathcal{H}$, and

$\sum_{h\in\mathcal{H}}x(\sigma;\sigma,h)+\sum_{h\in\mathcal{H}}x(\sigma;h,o)\le\omega(\sigma)$ for all $\sigma\in\Sigma_1$,

$\sum_{h\in\mathcal{H}}x(\sigma;\sigma,h)+\sum_{h\in\mathcal{H}}x(\sigma;\sigma_{-1},h)\le\omega(\sigma)$ for all $\sigma\in\cup_{t\ge2}\Sigma_t$.

We now introduce the notion of stationary equilibrium. Denote the stationary prices of the dividend paying asset by $q_s^{\mathrm{d}}\in R$ and for the jth inside asset by $q_s^j\in R$, both in state s. Let $q^{\mathrm{d}}:=(q_1^{\mathrm{d}},\cdots,q_S^{\mathrm{d}})$ and $q^j:=(q_1^j,\cdots,q_S^j)$, $j\in\mathcal{J}$, be the vectors of stationary prices of the assets and let $q_s:=(q_s^{\mathrm{d}},q_s^1,\cdots,q_s^J)$, $s\in\mathcal{S}$, be the vector of asset prices in state s. Given the nature of the problem, it is easy to see that the price of the commodity can be normalized to 1 at every node.

Stationarity of the equilibrium requires that $x(\sigma,h)=x(s(\sigma),h)$ for all $(\sigma,h)\in\Sigma\times\mathcal{H}$ (i.e., the consumption allocation of each agent depends on the state at the date of his birth and the states at the next date only); a stationary consumption plan for agent (σ,h), with $s=s(\sigma)$, will be denoted by $x(s,h)=(x(s;s,h),(x(s,s';s,h))_{s'\in\mathcal{S}})$ ($x(s;h,o)$ for the initial old). Stationary asset demands will be denoted by $\theta(s,h)=(\theta^{\mathrm{d}}(s,h),\theta^1(s,h),\cdots,\theta^J(s,h))$ for $(s,h)\in\mathcal{S}\times\mathcal{H}$. We also need to assign asset holdings to the initial old denoted $\theta(s;h,o)$; we allow the asset holding of the initial old to depend on the state of birth.

DEFINITION 2 (SCE):[4] $((x^*(s;h,o))_{s\in\mathcal{S},h\in\mathcal{H}},(\theta^*(s;h,o))_{s\in\mathcal{S},h\in\mathcal{H}},(x^*(s,h))_{s\in\mathcal{S},h\in\mathcal{H}},$ $(\theta^*(s,h))_{s\in\mathcal{S},h\in\mathcal{H}},q^{\mathrm{d}^*},(q^{j^*})_{j\in\mathcal{J}})$ is a *stationary competitive equilibrium with a sequence of markets (SCE)* if:

(i) x^* is a feasible stationary allocation;

(ii) for all $s\in\mathcal{S}$,

$\sum_{h\in\mathcal{H}}\theta^*(s;h,o)=(1,0,\cdots,0)$ and $\sum_{h\in\mathcal{H}}\theta^*(s,h)=(1,0,\cdots,0)$;

(iii) for all $s\in\mathcal{S}$, and for every $h\in\mathcal{H}$

(a) $x^*(s;h,o)\le\omega(s;h,o)+\theta^{\mathrm{d}^*}(s;h,o)[q_s^{\mathrm{d}^*}+d_s]+\sum_{j\in\mathcal{J}}\theta^{j^*}(s;h,o)r_s^j$;

(b) if $u_{s,h,o}(x)>u_{s,h,o}(x^*(s;h,o))$ then

$x>\omega(s;h,o)+\theta^{\mathrm{d}^*}(s;h,o)[q_s^{\mathrm{d}^*}+d_s]+\sum_{j\in\mathcal{J}}\theta^{j^*}(s;h,o)r_s^j$;

(iv) for all $s\in\mathcal{S}$, and for every $h\in\mathcal{H}$

(a) $x^*(s;s,h)+q_s^*\cdot\theta^*(s,h)\le\omega(s;s,h)$

$x^*(s,s';s,h)\le\omega(s,s';s,h)+\theta^{\mathrm{d}^*}(s,h)[q_{s'}^{\mathrm{d}^*}+d_{s'}]+\sum_{j\in\mathcal{J}}\theta^{j^*}(s,h)r_{s'}^j$ for all $s'\in\mathcal{S}$;

(b) if $u_{s,h}(x)>u_{s,h}(x^*(s,h))$ then either

$x(s;s,h)+q_s^*\cdot\theta^*(s,h)>\omega(s;s,h)$ or

$x(s,s';s,h)>\omega(s,s';s,h)+\theta^{\mathrm{d}^*}(s,h)[q_{s'}^{\mathrm{d}^*}+d_{s'}]+\sum_{j\in\mathcal{J}}\theta^{j^*}(s,h)r_{s'}^j$ for some $s'\in\mathcal{S}$.

REMARK 1: We have imposed the condition that all asset markets must clear exactly and have not imposed *free disposal* of asset prices. Also, the definition of stationary

[4]We ignore the existence of restrictions on the size of trades in the dividend paying assets.

equilibrium applies even when markets are *sequentially complete*, that is, if in every state the returns from the $1+J$ assets span R^S. We make no claims regarding existence; results on existence are available in certain special cases, e.g., when $d = \underline{0}$ so that the long-lived asset is money.

REMARK 2: The optimization problem solved by an agent can be written as

$$\max_{x,\theta} u_{s,h}(x, (\omega(s, s'; s, h) + \theta^{\mathrm{d}}[q^{\mathrm{d}*}_{s'} + d_{s'}] + \textstyle\sum_{j\in\mathcal{J}} \theta^j r^j_{s'})_{s'\in\mathcal{S}})$$
$$\text{subject to}: \quad x + q^*_s \cdot \theta \leq \omega(s; s, h).$$

So each agent, effectively, solves an optimization problem with a single budget constraint and will meet the constraint with equality. This property leads to the constrained optimality of all equilibria in two period economies.

3. THE UNIT ROOT PROPERTY

In this section we elucidate the relation between equilibrium prices and a certain number, the Perron root, which is associated with the optimization problem of each agent. This is important since it is known that when markets are sequentially complete the value of this number completely characterizes the optimality properties of the equilibrium allocation.

At an interior equilibrium allocaton, x^*, let $m_{s,s',h}(x^*) := \frac{du_{s,h}(x^*(s,h))}{dx_{1+s'}} / \frac{du_{s,h}(x^*(s,h))}{dx_1}$;[5] let $M_h(x^*)$ be the strictly positive square matrix with elements $m_{s,s',h}(x^*)$. By Perron's Theorem (see, e.g., Theorem 8.2.8 in Horn and Johnson (1985)), there exists a unique vector (up to normalization) with all components positive, $z_h \in R^S_{++}$, such that $[z_h]^T \cdot M_h = \lambda_h \cdot [z_h]^T$ for some number $\lambda_h \in R_{++}$. The number λ_h is the *Perron root* of the matrix M_h, and is the largest eigenvalue, in absolute value, of the matrix.

The next result shows that if dividends are nonnegative (and not always zero) then the Perron root exceeds one if and only if the price of the dividend paying asset is negative in some state. We are unable to say anything of interest about the value of the Perron root when dividends are allowed to be negative.

PROPOSITION 1: *Consider an equilibrium in which for some agent $h \in \mathcal{H}$, $x^*(s, h) \in R^{1+S}_{++}$ for all $s \in \mathcal{S}$. (i) If dividends are non-negative, $d \in R^S_+$, and assets are freely disposable and non-trivial, $q^{\mathrm{d}^*} \in R^S_+/\{\underline{0}\}$, then $\lambda_h \in (0, 1]$. (ii) If dividends are non-negative and non-trivial, $d \in R^S_+/\{\underline{0}\}$, and $q^{\mathrm{d}^*}_s < 0$ for some $s \in \mathcal{S}$, then $\lambda_h > 1$.*

PROOF: From the first order conditions for the optimal choice of $\theta^{\mathrm{d}^*}(s, h)$, as s varies, we obtain a set of matrix equations which must hold for all $h \in \mathcal{H}$ with $x^*(s, h) \in R^{1+S}_{++}$ for all $s \in \mathcal{S}$

$$q^{\mathrm{d}^*} = M_h \cdot (q^{\mathrm{d}^*} + d), \qquad q^{j^*} = M_h \cdot r^j.$$

Premultiplication of the equations for the long-lived asset by the vector $z_h \in R^S_{++}$ leads to

$$[z_h]^T \cdot q^{\mathrm{d}^*} = [z_h]^T \cdot M_h \cdot (q^{\mathrm{d}^*} + d) = \lambda_h \cdot [z_h]^T \cdot (q^{\mathrm{d}^*} + d)$$

[5] For $f : R^N_{++} \to R$, $\frac{df(\bar{x})}{dx_i}$ denotes the partial derivative of the function f with respect to its i-th coordinate evaluated at the point $\bar{x}$.

$$\Leftrightarrow \qquad \lambda_h \cdot [z_h]^T \cdot d = (1 - \lambda_h) \cdot [z_h]^T \cdot q^{\mathrm{d}^*} \tag{1}$$

where $\lambda_h > 0$.

(i) Clearly, if $d \in R^S_+$ and $q^{\mathrm{d}^*} \in R^S_+/\{\underline{0}\}$ then $\lambda_h \in (0, 1]$ as required.

(iia) Suppose that $q^{\mathrm{d}*}_s < 0$ for all $s \in \mathcal{S}$. Since, under the stated condition on dividends, $[z_h]^T \cdot d > 0$, and $\lambda_h > 0$, while $[z_h]^T \cdot q^{\mathrm{d}*} < 0$, we must have $\lambda_h > 1$, as required.

(iib) Now suppose that $q^{\mathrm{d}^*}_s \cdot q^{\mathrm{d}^*}_{s'} < 0$ for some $s, s' \in \mathcal{S}$, $s \neq s'$, so that $q^{\mathrm{d}^*}_s > 0$ for some $s \in \mathcal{S}$. Define $\tilde{q}_s := \max\{q^{\mathrm{d}^*}_s, 0\}$, for $s \in \mathcal{S}$, and construct the vector $\tilde{q} \in R^S_+/\{\underline{0}\}$; $\tilde{q} \neq \underline{0}$, since $q^{\mathrm{d}^*}_s > 0$ for some $s \in \mathcal{S}$, and $\tilde{q} - q^{\mathrm{d}^*} > \underline{0}$.[6] Since M_h is a strictly positive matrix we have (i) $M_h \cdot [\tilde{q} + d] >> \underline{0}$; also $M_h \cdot [q^{\mathrm{d}^*} + d] << M_h \cdot [\tilde{q} + d]$ so that the first order condition implies that (ii) $q^{\mathrm{d}^*} << M_h \cdot [\tilde{q} + d]$. Since for s such that $q^{\mathrm{d}^*}_s \geq 0$, $\tilde{q}_s = q^{\mathrm{d}^*}_s$, while for s such that $q^{\mathrm{d}^*}_s < 0$, $\tilde{q}_s = 0 > q^{\mathrm{d}^*}_s$, (i) and (ii) together imply that

$$\tilde{q} << M_h \cdot [\tilde{q} + d].$$

Premultiplying by the Perron vector, using the fact that $\tilde{q} > \underline{0}$, and using (1), we obtain

$$0 < [z_h]^T \cdot \tilde{q} < \lambda_h \cdot [z_h]^T \cdot \tilde{q} + \lambda_h \cdot [z_h]^T \cdot d = \lambda_h \cdot [z_h]^T \cdot \tilde{q} + (1 - \lambda_h) \cdot [z_h]^T \cdot q^{\mathrm{d}^*}$$

$$\Leftrightarrow \qquad [z_h]^T \cdot [\tilde{q} - q^{\mathrm{d}*}] < \lambda_h \cdot [z_h]^T \cdot [\tilde{q} - q^{\mathrm{d}*}].$$

As we noted earlier, $\tilde{q} - q^{\mathrm{d}^*} > \underline{0}$; it follows that $\lambda_h > 1$. ∎

A natural question concerns the possibility of obtaining equilibria in which the price of the dividend paying asset is negative in some state even though the dividend is always non-negative. The example that follows gives an affirmative answer.

EXAMPLE 1:[7] Let there be one good, two Markov states, $\{s_a, s_b\}$, with transition probabilities $\pi_{s_a,s_a} = \pi_{s_b,s_b} = 0.75$, and $d(s_a) = 1$ while $d(s_b) = 0.5$. There is one agent with preferences described by $u(c_s, c_{s,s_a}, c_{s,s_b}) = \ln c_s + \pi_{s,s_a} \ln c_{s,s_a} + \pi_{s,s_b} \ln c_{s,s_b}$ and endowments $(\omega_{s_a}, \omega_{s_a,s_a}, \omega_{s_a,s_b}) = (146/7, 6, 3)$ and $(\omega_{s_b}, \omega_{s_b,s_a}, \omega_{s_b,s_b}) = (91/22, 6, 3)$. It may be verified that $q_{s_a} = -2$ and $q_{s_b} = .5$ constitute equilibrium prices.

If the market is sequentially complete, i.e., $J \geq S - 1$ and the assets span R^S, the matrix M_h is necessarily the same for all the agents and there is an unambiguous value for λ. The equilibrium is said to exhibit the *unit root property* if the Perron root is less than or equal to one.

When the market fails to be sequentially complete, e.g., because $J < S - 1$, the matrices M_h typically differ across agents. Proposition 1, however, continues to apply.

[6] When comparing two vectors x and y of the same dimension we use the symbols "$\leq$", "$<$", and "$<<$" to indicate $x_n \leq y_n$ for all n, $x_n \leq y_n$ for all n but $x \neq y$, and $x_n < y_n$ for all n respectively.

[7] This example builds on an example of a deterministic economy that appeared in an earlier version of Santos and Woodford (1997). They attribute it to W. Brock, the probable reference being Brock (1990).

4. CONSTRAINED OPTIMALITY

We want to analyze the optimality properties of the stationary equilibria that we have defined. We begin by introducing a notion of optimality which applies the criterion of Pareto efficiency to the economy above where agents are distinguished by the event at their birth. This yields the criterion of conditional Pareto Optimality, due to Muench (1977):

DEFINITION 3 (CPO): Let x be a feasible allocation. x is *conditionally Pareto optimal (CPO)* if there does not exist another feasible allocation $\widehat{x}$ such that

(i) for all $(\sigma, h) \in \Sigma_1 \times \mathcal{H}$, $u_{s(\sigma),h,o}(\widehat{x}(\sigma; h, o)) \geq u_{s(\sigma),h,o}(x(\sigma; h, o))$,
for all $(\sigma, h) \in \Sigma \times \mathcal{H}$, $u_{s(\sigma),h}(\widehat{x}(\sigma, h)) \geq u_{s(\sigma),h}(x(\sigma, h))$;

(ii) either for some $(\sigma', h') \in \Sigma_1 \times \mathcal{H}$, $u_{s(\sigma'),h',o}(\widehat{x}(\sigma'; h', o)) > u_{s(\sigma'),h',o}(x(\sigma'; h', o))$,
or for some $(\sigma', h') \in \Sigma \times \mathcal{H}$, $u_{s(\sigma'),h'}(\widehat{x}(\sigma', h')) > u_{s(\sigma'),h'}(x(\sigma', h'))$.

It is by now well known that a stationary equilibrium allocation obtained with sequentially complete markets is CPO if and only if the unit root property holds (see Aiyagari and Peled (1991), Chattopadhyay and Gottardi (1999), and Demange and Laroque (1999)).[8]

When markets fail to be sequentially complete, the equilibrium allocation is typically not CPO. The argument is the same as the one used in two period economies, i.e., the vectors of marginal rates of substitution differ across agents. What constitutes an appropriate test of efficiency in this case is a problematic point. Here we use a definition that requires the planner to allocate resources using the existing assets and holding asset prices at their equilibrium value. Recall that $\sum_{h\in\mathcal{H}} \omega^{\mathrm{d}}(s; h, o) = 1$ so that only one unit of the dividend paying asset is available in the economy.

DEFINITION 4 (q^*-CF): A feasible allocation $\widehat{x}$ is q^*-*constrained feasible* if there exist $((\widehat{\theta}^{\mathrm{d}}(s; h, o))_{s\in\mathcal{S},h\in\mathcal{H}})$, $(((\widehat{\theta}^{j}(s; h, o))_{s\in\mathcal{S},h\in\mathcal{H}})_{j\in\mathcal{J}})$, $((\widehat{\theta}^{\mathrm{d}}(\sigma, h))_{\sigma\in\Sigma,h\in\mathcal{H}})$, $(((\widehat{\theta}^{j}(\sigma, h))_{\sigma\in\Sigma,h\in\mathcal{H}})_{j\in\mathcal{J}})$) such that:

(ia) for all $s \in \mathcal{S}$, $\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(s; h, o) \leq 1$ and $\sum_{h\in\mathcal{H}} \widehat{\theta}^{j}(s; h, o) = 0$ for all $j \in \mathcal{J}$,

(ib) for all $\sigma \in \Sigma$, $\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(\sigma, h) \leq 1$ and $\sum_{h\in\mathcal{H}} \widehat{\theta}^{j}(\sigma, h) = 0$ for all $j \in \mathcal{J}$;

(iia) for all $h \in \mathcal{H}$ and $s \in \mathcal{S}$,
$\widehat{x}(s; h, o) = \omega(s; h, o) + \widehat{\theta}^{\mathrm{d}}(s; h, o)[q_s^{\mathrm{d}*} + d_s] + \sum_{j\in\mathcal{J}} \widehat{\theta}^{j}(s; h, o) r_s^j$;

(iib) for all $h \in \mathcal{H}$ and $\sigma \in \Sigma$,
$\widehat{x}(\sigma, s'; \sigma, h) = \omega(\sigma, s'; \sigma, h) + \widehat{\theta}^{\mathrm{d}}(\sigma, h)[q_{s'}^{\mathrm{d}*} + d_{s'}] + \sum_{j\in\mathcal{J}} \widehat{\theta}^{j}(\sigma, h) r_{s'}^j$ for all $s' \in \mathcal{S}$.

DEFINITION 5 (q^*-CF): An allocation $\widehat{x}$ is a q^*-*constrained improvement* if it is q^*-*constrained feasible* and a CPO improvement.

---

[8] The result in Chattopadhyay and Gottardi (1999) provides a complete characterization of those competitive equilibria that are CPO and that can be obtained via trade in contingent commodity markets. The kind of equilibrium constructed in Example 1 cannot be obtained via trade in contingent commodity markets so that the unit root characterization does not apply to it. However, Chattopadhyay (2003) provide a characterization result which covers such cases.

We now provide a condition under which a stationary constrained improvement can be constructed. The condition is a generalization of the configuration of prices and dividends in Brock's deterministic example since it requires that the equilibrium price of the long lived asset be negative in sign in some state.[9]

PROPOSITION 2: *Consider an interior stationary equilibrium with sequential markets. Let* $\bar{\mathcal{S}} := \{s \in \mathcal{S} : q_s^{\mathrm{d}^*} + d_s < 0\}$. *If* $\bar{\mathcal{S}} \neq \emptyset$ *and* $d_s > 0$ *for all* $s \in \bar{\mathcal{S}}$, *then there exists a stationary* q^**-constrained improvement.*

PROOF: We construct a q^*-feasible allocation which CPO improves over the equilibrium allocation.

For each $s \in \bar{\mathcal{S}}$, define $\Delta(s) := -\frac{d_s}{(c+1)[q_s^{\mathrm{d}^*}+d_s]^2}$, where $c > 0$ is an upper bound on the curvature of the agents' indifference surfaces. c is well defined given Assumption 1 (since differentiability and strict monotonicity of the utility function implies that each agent's indifference surface has bounded curvature at each point) and the fact that a stationary allocation is specified by a finite set of vectors (one for each type of agent born in each of S different states). Let $\Delta := -\min_{s\in\bar{\mathcal{S}}}|\Delta(s)|$. So $\Delta < 0$. Choose an agent born in each of the Markov states $s \in \bar{\mathcal{S}}$, denoted $(s, \bar{h})$, and set $\widehat{\theta}^{\mathrm{d}}(s, \bar{h}) := \theta^{\mathrm{d}^*}(s, \bar{h}) + \Delta$ at every date. Also set $\widehat{x}(s; s, \bar{h}) := x^*(s; s, \bar{h}) + [q_s^{\mathrm{d}^*} + d_s](-1)\Delta$. Do not change any other variables for the initial old or agents born in states not in $\bar{\mathcal{S}}$. Clearly, in states faced by $(s, \bar{h})$ when old at which $[q_{s'}^{\mathrm{d}^*} + d_{s'}] > 0$, i.e., when $s' \in \mathcal{S}\backslash\bar{\mathcal{S}}$, agent $(s, \bar{h})$ consumes *less* of the good relative to the equilibrium allocation (and this amount is available as slack) while he consumes more in the states $s \in \bar{\mathcal{S}}$. At each such state we have a young agent who consumes less, thus maintaining feasibility, and whose portfolio assignment has been changed. The marginal change in utility experienced by the agent when old, normalized by the marginal utility of consumption when young, due to the change in his portfolio can be computed from the first order condition and is given by $q_s^{\mathrm{d}^*}\Delta$, while the (normalized) change due to the change in consumption when young is $[q_s^{\mathrm{d}^*} + d_s](-1)\Delta$. Summing the two components, and using the definition of Δ, we get the net marginal change in the utility value of the agent's allocation, $d_s(-1)\Delta > 0$. In order to ensure that the agent $(s, \bar{h})$ is being improved, it is sufficient that the net marginal change in the utility value of the allocation satisfy the following quadratic inequality where c is as specified earlier, i.e., a uniform upper bound on the curvature of the agents' indifference sets:[10]

$$d_s(-1)\Delta \geq c([q_s^{\mathrm{d}^*} + d_s](-1)\Delta)^2.$$

Since $|\Delta| \leq |\Delta(s)|$ and the inequality is specified by a quadratic function, it suffices to show that $\Delta(s)$ satisfies the inequality for every $s \in \bar{\mathcal{S}}$. To see this, notice that

$$d_s(-1)\Delta(s) = \frac{[d_s]^2}{(c+1)[q_s^{\mathrm{d}^*}+d_s]^2}.$$

Also,

$$c([q_s^{\mathrm{d}^*} + d_s](-1)\Delta(s))^2 = \frac{1}{(c+1)^2}c\frac{[d_s]^2}{[q_s^{\mathrm{d}^*}+d_s]^2}.$$

[9]By writing relative prices in terms of discounted prices of the commodity one shows that the equilibrium in Brock's example is inflationary which is generally indicative of suboptimality.

[10]See, e.g., Lemma 1 in Chattopadhyay and Gottardi (1999).

So,

$$d_s(-1)\Delta(s) \geq c([q_s^{\mathrm{d}^*} + d_s](-1)\Delta(s))^2$$

as required. ■

COROLLARY 1: *With sequentially incomplete markets and strictly positive dividends, a stationary equilibrium allocation at which the Perron root of some agent's matrix of marginal rates of substitution exceeds one can be q^*-constrained CPO improved.*

The proof follows by noting that under the stated condition on the Perron root, Proposition 1 implies that the price of the dividend paying asset must be negative in some state. But then the first order conditions for optimal choice of the dividend paying asset imply that the set $\bar{\mathcal{S}}$ specified in Proposition 2 is non-empty. Hence, the fact that dividends are strictly positive allows us to invoke Proposition 2 to complete the proof.

We turn to sufficient conditions under which the stationary equilibrium allocation is q^*-constrained CPO.

PROPOSITION 3: *Consider a stationary equilibrium with sequential markets. If $q_s^{\mathrm{d}^*} + d_s \geq 0$ for all $s \in \mathcal{S}$, $q_{\tilde{s}}^{\mathrm{d}^*} + d_{\tilde{s}} > 0$ for some $\tilde{s} \in \mathcal{S}$, then the allocation is q^*-constrained CPO.*

PROOF: Under the stated hypotheses, $q_s^{\mathrm{d}^*} > 0$ for all $s \in \mathcal{S}$; this follows from the first order conditions.

Let $\widehat{x}$ denote a q^*-constrained feasible allocation. For $\sigma \in \Sigma$ and $h \in \mathcal{H}$, define

$$\Delta\widehat{x}(\sigma;\sigma,h) := \widehat{x}(\sigma;\sigma,h) - x^*(\sigma;\sigma,h).$$

Using Definition 4 together with the budget constraints specified in Definition 2, and the equilibrium market clearing conditions for assets, we obtain:

$$\textstyle\sum_{h\in\mathcal{H}} \Delta\widehat{x}(s(\sigma);h,o) = [\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(s(\sigma);h,o) - 1] \cdot [q_{s(\sigma)}^{\mathrm{d}^*} + d_{s(\sigma)}] \qquad \text{for all } \sigma \in \Sigma_1,$$

$$\textstyle\sum_{h\in\mathcal{H}} \Delta\widehat{x}(\sigma,s';\sigma,h) = [\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(\sigma,h) - 1] \cdot [q_{s'}^{\mathrm{d}^*} + d_{s'}] \qquad \text{for all } \sigma \in \Sigma \text{ and } s' \in \mathcal{S}.$$

These restrictions delimit the net changes in consumption when old by the requirement that they be asset market feasible. By combining these restrictions with the aggregate feasibility condition specified in Definition 1, and noting that at equilibrium the aggregate feasibility condition holds with equality, we obtain

$$\textstyle\sum_{h\in\mathcal{H}} \Delta\widehat{x}(\sigma;\sigma,h) + [\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(s(\sigma);h,o) - 1] \cdot [q_{s(\sigma)}^{\mathrm{d}^*} + d_{s(\sigma)}] \leq 0 \qquad \text{for all } \sigma \in \Sigma_1 \quad \text{(2a)}$$

$$\textstyle\sum_{h\in\mathcal{H}} \Delta\widehat{x}(\sigma,s';\sigma,s',h) + [\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(\sigma,h) - 1] \cdot [q_{s'}^{\mathrm{d}^*} + d_{s'}] \leq 0 \qquad \text{for all } \sigma \in \Sigma \text{ and } s' \in \mathcal{S}. \quad \text{(2b)}$$

Now suppose that the allocation $\widehat{x}$ is also a CPO improvement. Since agents are locally non-satiated (this follows trivially for the initial old and follows from Remark 2 for the rest of the agents), the improving allocation must be at least as costly as the equilibrium allocation for every agent and strictly more costly for some agent. So, considering the initial old and summing over the set of agents, the following inequality must hold:

$$\textstyle[\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(s(\sigma);h,o) - 1] \cdot [q_{s(\sigma)}^{\mathrm{d}^*} + d_{s(\sigma)}] \geq 0 \qquad \text{for all } \sigma \in \Sigma_1, \quad \text{(3a)}$$

while for the rest of the agents

$$\textstyle\sum_{h\in\mathcal{H}} \Delta\widehat{x}(\sigma;\sigma,h) + q^{\mathrm{d}^*}_{s(\sigma)} \cdot [\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(\sigma,h) - 1] \geq 0 \qquad \text{for all } \sigma \in \Sigma \tag{3b}$$

with strict inequality for some $\sigma \in \Sigma$.

The necessary condition for improving the initial old, (3a), together with the feasibility condition on allocating the dividend paying asset, $\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(s;h,o) \leq 1$, and the hypothesis of Proposition 3 imply that $[\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(s(\sigma);h,o) - 1] \cdot [q^{\mathrm{d}^*}_{s(\sigma)} + d_{s(\sigma)}] = 0 \qquad \text{for all } \sigma \in \Sigma_1$. It follows from (2a) that $\sum_{h\in\mathcal{H}} \Delta\widehat{x}(\sigma;\sigma,h) \leq 0$ for all $\sigma \in \Sigma_1$. But then (3b) together with $\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(\sigma,h) \leq 1$ implies that $\sum_{h\in\mathcal{H}} \widehat{\theta}^{\mathrm{d}}(\sigma,h) = 1$ provided that $q^{\mathrm{d}^*}_{s(\sigma)} > 0$ which holds as noted at the beginning of the proof. But now an iterative argument, node by node, shows that there can be no q^*-constrained feasible improvement since (3b) will always hold with equality. ■

COROLLARY 2: *With sequentially incomplete markets, non-negative and non-trivial dividends, and free disposal of the asset, all stationary equilibrium allocations are q^*-constrained CPO.*

The proof of Corollary 1 follows directly from Proposition 3 by noting that under the stated conditions the price of the asset is necessarily positive in every state.

5. DISCUSSION

The nature of the relationship between the unit root property and optimality when markets fail to be sequentially complete can now be addressed. Corollaries 1 and 2 have shown that when the dividend vector is strictly positive the unit root condition is necessary and sufficient to determine the optimality properties of a stationary equilibrium. The relation between the unit root property and optimality breaks down when the dividend vector is non-negative but with some zero entries; simple examples exist in which, for some agent, the Perron root exceeds one, so that the price of the asset is negative in some states, but these are states in which the dividend is zero so that Proposition 2 does not apply. Very little can be said when the dividend vector has negative components as Proposition 1 does not extend to such cases.

So the relation between the unit root condition and optimality does survive into the domain of sequentially incomplete markets provided that dividends are strictly positive; this extension of the unit root property is undermined by the fact that under the stated conditions and the additional requirement that assets be freely disposable, which seems reasonable when dividends are positive, every equilibrium allocation is constrained CPO. The relation is less clear when more general specifications are allowed for the dividend process as well as prices.

The extent to which the results change when we require the improving allocation to be obtainable as an equilibrium is of interest and a challenging open question.

REFERENCES

AIYAGARI, S. R., AND D. PELED (1991): "Dominant Root Characterization of Pareto Optimality and the Existence of Optimal Monetary Equilibria in Stochastic OLG Models," *Journal of Economic Theory*, 54, 69-83.

ALLEN, F., AND D. GALE (1997): "Financial Markets, Intermediaries, and Intertemporal Smoothing," *Journal of Political Economy*, 105, 523-546.

BROCK, W. (1990): "Overlapping Generations Models," in "The Handbook of Monetary Economics" Vol. I, eds. Hahn, F., and Friedman, B. (North Holland).

CASS, D., R. GREEN, AND S. SPEAR (1992): "Stationary Equilibria with Incomplete Markets and Overlapping Generations," *International Economic Review*, 33, 495-512.

CHATTOPADHYAY, S. (2003): "Stochastic OLG Models, Asset Markets, and Optimality."

CHATTOPADHYAY, S., AND P. GOTTARDI (1999): "Stochastic OLG Models, Market Structure, and Optimality," *Journal of Economic Theory*, 89, 21-67.

DEMANGE, G. (2002): "On Optimality in Intergenerational Risk Sharing," *Economic Theory*, 20, 1-27.

DEMANGE, G., AND G. LAROQUE (1999): "Social Security and Demographic Shocks," *Econometrica*, 67, 527-542.

GEANAKOPLOS, J. D., AND H. M. POLEMARCHAKIS (1986): "Existence, Regularity, and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete," in "Essays in Honour of K. J. Arrow" Vol. III, eds. Heller, W., Starrett, D., and Starr, R. (Cambridge).

GOTTARDI, P. (1996): "Constrained Efficiency of Stationary Monetary Equilibria in Overlapping Generations Models with Incomplete Markets" (mimeo.).

HORN, R. A., AND C. R. JOHNSON (1985):"Matrix Analysis," Cambridge University Press, Cambridge.

MUENCH, T. J. (1977): "Optimality, the Interaction of Spot and Futures Markets, and the Non-neutrality of Money in the Lucas Model," *Journal of Economic Theory*, 15, 325-344.

SANTOS, M., AND M. WOODFORD (1997): "Rational Asset Pricing Bubbles," *Econometrica*, 65, 19-57.

SCHEINKMAN, J. A. (1980): "Notes on Asset Trading in OLG Economies," (mimeo.).

WILSON, C. (1981): "Equilibrium in Dynamic Models with an Infinity of Agents," *Journal of Economic Theory*, 24, 95-111.